This video will teach us how to solve a linear second-order differential equation in homogenous and non-homogenous formats. In the latter, the answer is a polynomial.

$a.y'' + b.y' + c.y = 0$ as a homogenous equation

$a.y'' + b.y' + c.y = a_0 + a_1.x + a_2.x^2 + \cdots + a_n.x^n$ as a non-homogenous polynomial equation

Moreover, some tips for the calculation of stability equations are presented in the following format:

$y'' + \lambda^2.y = k.x$

$f(x)\,y'' + g(x)\,y' + h(x)\,y = I(x)$

$a y'' + b y' + c y = I(x)$

$I(x) = 0 \Rightarrow$ Homogenous equation $\rightarrow a y'' + b y' + c y = 0$
(characteristic equation)

$I(x) \neq 0 \Rightarrow$ Non-homogenous equation

Solution of homogenous equation:

$a r^2 + b r + c = 0 \Rightarrow \boxed{r_1, r_2}$ $\quad \Delta = b^2 - 4 \cdot a \cdot c$

$\rightarrow r_1 \neq r_2 \in \mathbb{R} \Rightarrow y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$r = r_1 = r_2 \ , (\Delta = 0) \Rightarrow y_h = c_1 e^{rt} + c_2 t \cdot e^{rt}$

$r_1 = \alpha + i\beta$
$r_2 = \alpha - i\beta$ $\Rightarrow y_h = e^{\alpha t}\left[c_1 \sin \beta t + c_2 \cos \beta t\right]$
$\Delta < 0 \ , \ i = \sqrt{-1}$

$y'' - 2y' - 3 = 0$

$r^2 - 2r - 3 = 0 \ , \ \Delta = (-2)^2 - 4(1)(-3) = 4 + 12 = 16 \Rightarrow r = \frac{2 \pm \sqrt{16}}{2} = \begin{cases} 3 \\ -1 \end{cases}$

$\begin{cases} r_1 = 3 \\ r_2 = -1 \end{cases} \rightarrow y_h = c_1 \cdot e^{3t} + c_2 \cdot e^{-t}$

$$y'' + 2y' + 1 = 0 \qquad r^2 + 2r + 1 = 0 \Rightarrow r = \frac{-2 \pm 0}{2} = -1$$

$$\Delta = (2)^2 - 4(1)(1) = 0$$

$$(r+1)^2 = 0 \Rightarrow r_1 = r_2 = -1$$

$$y_h = c_1 e^{-t} + c_2 t \cdot e^{-t}$$

$$y'' + 2y' + 2y = 0 \Rightarrow r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{-4}}{2}$$

$$\Delta = (2)^2 - 4(1)(2) = -4$$

$$r = \frac{-2 \pm 2\sqrt{-1}}{2} = -1 \pm \underbrace{\sqrt{-1}}_{i} = -1 \pm i \qquad \begin{cases} \alpha = -1 \\ \beta = 1 \end{cases}$$

$$y_h = e^{-t}\left[c_1 \sin t + c_2 \cos t\right]$$

$$y'' + 4y = 0 \qquad r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow$$

$$r = \pm\sqrt{-4} = \pm 2\sqrt{-1} = \boxed{\pm 2i}$$

$$\begin{cases} \alpha = 0 \\ \beta = 2 \end{cases} \Rightarrow y_h = \underbrace{e^{0t}}_{0}\left[c_1 \sin 2t + c_2 \cos 2t\right]$$

$$\boxed{y_h = c_1 \sin 2t + c_2 \cos 2t}$$

$$y'' + \lambda^2 y = 0 \Rightarrow r^2 + \lambda^2 = 0 \Rightarrow r^2 = -\lambda^2 \Rightarrow r = \pm\sqrt{-\lambda^2} = \pm\lambda\sqrt{-1} = \pm i\lambda$$

$$\left.\begin{matrix} \alpha = 0 \\ \beta = \lambda \end{matrix}\right\} \rightarrow y_h = A \sin \lambda t + B \cos \lambda t$$

non homogenous form: (linear condition)

$$ay'' + by' + c = I(x)$$

$$I(x) = a_0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$$

① solve the homogenous form of the equation. $\rightarrow y_h$

② Find particular solution relevant to $I(x)$ $\rightarrow y_p$

③ $Y = y_h + y_p$

$$y_p = x^s \left[A_0 + A_1 x^1 + A_2 x^2 + \cdots\right]$$

$s = ? \rightarrow$ s is the number of zero solution for characteristic equation.

$s = 0, 1, 2$

Example: $y'' - 2y' - 3y = t^2$ (I)

① $y'' - 2y' - 3y = 0$, $r^2 - 2r - 3 = 0 \Rightarrow r = \dfrac{2 \pm \sqrt{4 + 12}}{2} = \begin{cases} 3 \\ -1 \end{cases}$

$$y_h = c_1 e^{3t} + c_2 e^{-t}$$

② How many time zero is the root of characteristic equation $\begin{cases} r_1 = 3 \\ r_2 = -1 \end{cases} \Rightarrow$ No zero $\rightarrow \boxed{S = 0}$

$$y_p = \underset{S=0}{t^s}\left[At^2 + Bt + C\right] = At^2 + Bt + C$$

$y_p' = 2At + B$, $y_p'' = 2A$

$$2A - 2(2At + B) - 3(At^2 + Bt + C) = t^2$$

$$2A - 4At - 2B - 3At^2 - 3Bt - 3C = t^2$$

$$\underline{-3A}\,t^2 + (\underline{-4A - 3B})t + (\underline{2A - 2B - 3C}) = \underline{t^2} \; \underline{0t} \; \underline{0}$$

$-3A = 1 \rightarrow \boxed{A = -\frac{1}{3}}$

$-4A - 3B = 0 \Rightarrow 3B = -4A = \frac{4}{3} \Rightarrow \boxed{B = \frac{4}{9}}$

$2A - 2B - 3C = 0$

$\Rightarrow 2\left(-\frac{1}{3}\right) - 2\left(\frac{4}{9}\right) = 3C$

$-\frac{2}{3} - \frac{8}{9} = 3C \Rightarrow \boxed{C = -\frac{14}{27}}$

$$\begin{cases} y_p = -\frac{1}{3}t^2 + \frac{4}{9}t - \frac{14}{27} \\ y' = -\frac{2}{3}t + \frac{4}{9} \\ y'' = -\frac{2}{3} \end{cases}$$

$$\begin{cases} y_p = -\frac{1}{3}t^2 + \frac{4}{9}t - \frac{14}{27} \\ y' = -\frac{2}{3}t + 4/9 \\ y'' = -\frac{2}{3} \end{cases}$$

$$y'' - 2y' - 3y = 0$$

$$-\frac{2}{3} - 2\left(-\frac{2}{3}t + \frac{4}{9}\right) - 3\left(-\frac{1}{3}t^2 + \frac{4}{9}t - \frac{14}{27}\right)$$

$$= \underbrace{-\frac{2}{3} + \cancel{\frac{4}{3}t} - \frac{8}{9}}\ \underbrace{+ t^2 - \cancel{\frac{4}{3}t} + \frac{14}{9}} = \boxed{t^2}$$

$$\frac{-6-8+14}{9} = 0$$

$$Y = y_h + y_p = c_1 e^{3t} + c_2 e^{-t} - \frac{1}{3}t^2 + \frac{4}{9}t - \frac{14}{27}$$

Example:

$$y'' + \lambda^2 y = k \cdot x$$

① $y'' + \lambda^2 y = 0 \Rightarrow y_h = A \sin\lambda x + B \cos\lambda x$,

② $y_p = x^0 \left[c_1 x + c_2\right] \Rightarrow \begin{cases} y_p = c_1 x + c_2 \\ y' = c_1 \\ y'' = 0 \end{cases}$

$$\lambda^2 \cdot y = \lambda^2 (c_1 x + c_2) = kx \Rightarrow c_1 x + c_2 = \frac{k}{\lambda^2} \cdot x$$

$$c_2 = 0 \quad , \quad c_1 = \frac{k}{\lambda^2} \cdot x$$

$$\Rightarrow \boxed{y_p = \frac{k}{\lambda^2} \cdot x}$$

$$y'' + \lambda^2 y = k \cdot x \Rightarrow \boxed{y = A \sin\lambda x + B \cos\lambda x + \frac{k}{\lambda^2} \cdot x}$$

Example:

$$y'' + 4y' = x^2 + x \quad (\mathrm{I})$$

$$r^2 + 4r = 0 \Rightarrow \begin{cases} r = 0 \\ r = -4 \end{cases} \qquad y_h = c_1 e^{0x} + c_2 e^{-4x} = c_1 + c_2 e^{-4x}$$

$$r = 0 \rightarrow \boxed{s=1} \qquad y_p = x^1\left[Ax^2 + Bx + c\right] = Ax^3 + Bx^2 + cx$$

$$y' = 3Ax^2 + 2Bx + C$$

$$y'' = 6Ax + 2B$$

$$(\mathrm{I}) \Rightarrow (6Ax + 2B) + 4(3Ax^2 + 2Bx + c) = x^2 + x$$

$$6Ax + 2B + 12Ax^2 + 8Bx + 4c = x^2 + x$$

$$12Ax^2 + (6A + 8B)x + (2B + 4c) = x^2 + x$$

$$12A = 1 \rightarrow A = \frac{1}{12}$$

$$6A + 8B = 1 \qquad \frac{1}{2} + 8B = 1 \rightarrow 8B = \frac{1}{2} \Rightarrow B = \frac{1}{16}$$

$$2B + 4c = 0 \qquad c = -\frac{B}{2} = -\frac{1}{32}$$

$$y_p = \frac{1}{12}x^3 + \frac{1}{16}x^2 - \frac{1}{32}x, \quad y' = \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{32}, \quad y'' = \frac{1}{2}x + \frac{1}{8}$$

$$y'' + 4y' = \underline{\underline{x^2 + x}} \Rightarrow \left(\frac{1}{2}x + \frac{1}{8}\right) + 4\left(\frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{32}\right) =$$

$$\underbrace{\frac{1}{2}x + \cancel{\frac{1}{8}} + x^2 + \frac{1}{2}x}_{x} - \cancel{\frac{1}{8}} = x^2 + x$$

$$Y = y_h + y_p = c_1 + c_2 e^{-4x} + \frac{1}{12}x^3 + \frac{1}{16}x^2 - \frac{1}{32}x$$