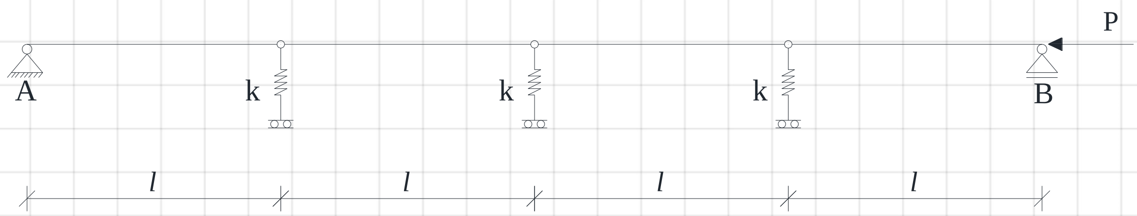
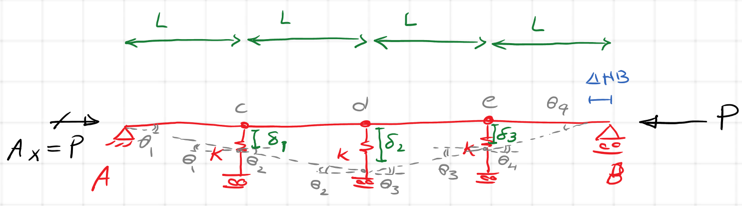
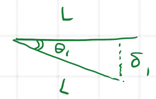
Four rigid bars form a compressive element under applied compressive load at point B. It is assumed that the bars are hinged and connected at their intersection. At one end, the element is supported by a hinge, and at the other end, point B, the element is supported by a roller. The system is assumed to be a 2D structure. At the intersections, there are three transitional springs with a spring coefficient of k. Each rigid bar is the length of l.

a) Determine the total potential energy of the system.
b) Determine the buckling loads.
c) Determine the buckling shape modes.
d) If the undrained shear strength of soil is $10kPa$, determine the buckling load numerically.
e) For three Circular Hollow Sections (CHS[1]), determine the Euler buckling load and check which buckling mode would occur first.
f) Model the system in FEM software and compare your results.



[1] CHS88,9/6,3, CHS139,7/6,3, and CHS168,3/10





$$\left.\begin{array}{l}\sin\theta_1 \simeq \theta_1 \\ \theta_1 \to 0 \\ \sin\theta_1 = \dfrac{\delta_1}{L}\end{array}\right\} \Rightarrow \theta_1 = \frac{\delta_1}{L}$$

$$\theta_2 = \frac{\delta_2 - \delta_1}{L}, \quad \theta_3 = \frac{\delta_2 - \delta_3}{L}, \quad \theta_4 = \frac{\delta_3}{L}$$

$$\cos\theta \simeq 1 - \frac{\theta^2}{2} \Rightarrow 1 - \cos\theta = \frac{\theta^2}{2}$$

$$\theta \to 0$$

$$\Delta HB = L - L\cos\theta_1 + L - L\cos\theta_2 + L - L\cos\theta_3 + L - L\cos\theta_4$$

$$\Delta HB = L\left[1 - \cos\theta_1 + 1 - \cos\theta_2 + 1 - \cos\theta_3 + 1 - \cos\theta_4\right]$$

$$\Delta HB = L\left[\frac{\theta_1^2}{2} + \frac{\theta_2^2}{2} + \frac{\theta_3^2}{2} + \frac{\theta_4^2}{2}\right]$$

$$\Delta HB = \frac{L}{2}\left[\left(\frac{\delta_1}{L}\right)^2 + \left(\frac{\delta_2 - \delta_1}{L}\right)^2 + \left(\frac{\delta_2 - \delta_3}{L}\right)^2 + \left(\frac{\delta_3}{L}\right)^2\right]$$

$$\Delta HB = \frac{1}{2L}\left[\delta_1^2 + \delta_2^2 + \delta_1^2 - 2\delta_1\delta_2 + \delta_2^2 + \delta_3^2 - 2\delta_2\delta_3 + \delta_3^2\right]$$

$$\Delta HB = \frac{1}{L}\left[\delta_1^2 + \delta_2^2 + \delta_3^2 - \delta_1\delta_2 - \delta_2\delta_3\right]$$

$$V = -P\cdot\Delta HB = -\frac{P}{L}\left[\delta_1^2 + \delta_2^2 + \delta_3^2 - \delta_1\delta_2 - \delta_2\delta_3\right]$$

$$W = \sum \frac{1}{2} k\cdot x^2 = \frac{1}{2}\left[k\delta_1^2 + k\delta_2^2 + k\delta_3^2\right] = \frac{k}{2}\left[\delta_1^2 + \delta_2^2 + \delta_3^2\right]$$

$$\Pi(\delta_1, \delta_2, \delta_3) = W + V = \frac{k}{2}\left[\delta_1^2 + \delta_2^2 + \delta_3^2\right] - \frac{P}{L}\left[\delta_1^2 + \delta_2^2 + \delta_3^2 - \delta_1\delta_2 - \delta_2\delta_3\right]$$

$$\frac{\partial\Pi}{\partial\delta_1} = 0, \quad \frac{\partial\Pi}{\partial\delta_2} = 0, \quad \frac{\partial\Pi}{\partial\delta_3} = 0$$

$$\frac{\partial\Pi}{\partial\delta_1} = \frac{k}{2}\left[2\delta_1\right] - \frac{P}{L}\left[2\delta_1 - \delta_2\right] = 0 \Rightarrow$$

$$\frac{\partial\Pi}{\partial\delta_2} = \frac{k}{2}\left[2\delta_2\right] - \frac{P}{L}\left[2\delta_2 - \delta_1 - \delta_3\right] = 0 \Rightarrow$$

$$\frac{\partial\Pi}{\partial\delta_3} = \frac{k}{2}\left[2\delta_3\right] - \frac{P}{L}\left[2\delta_3 - \delta_2\right] = 0 \Rightarrow$$

$$\left\{\begin{array}{l}\left(k - \dfrac{2P}{L}\right)\delta_1 + \dfrac{P}{L}\delta_2 + 0\,\delta_3 = 0 \\ \dfrac{P}{L}\delta_1 + \left(k - \dfrac{2P}{L}\right)\delta_2 + \dfrac{P}{L}\delta_3 = 0 \\ 0\,\delta_1 + \dfrac{P}{L}\delta_2 + \left(k - \dfrac{2P}{L}\right)\delta_3 = 0\end{array}\right.$$

$$\begin{cases} \left(k-\frac{2P}{L}\right)\delta_1 + \frac{P}{L}\delta_2 + 0\,\delta_3 = 0 \\ \frac{P}{L}\delta_1 + \left(k-\frac{2P}{L}\right)\delta_2 + \frac{P}{L}\delta_3 = 0 \\ 0\,\delta_1 + \frac{P}{L}\delta_2 + \left(k-\frac{2P}{L}\right)\delta_3 = 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} k-\frac{2P}{L} & \frac{P}{L} & 0 \\ \frac{P}{L} & k-\frac{2P}{L} & \frac{P}{L} \\ 0 & \frac{P}{L} & k-\frac{2P}{L} \end{bmatrix}}_{A} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \bar{0}$$

$\det(A) = 0 \Rightarrow$

$$\det(A(k,p,l)) \xrightarrow[\textit{expand}]{\textit{simplify}} -\frac{4\cdot p^3}{l^3} + \frac{10\cdot k\cdot p^2}{l^2} - \frac{6\cdot k^2\cdot p}{l} + k^3 = 0$$

$$-4\left(\frac{P}{kL}\right)^3 + 10\left(\frac{P}{kL}\right)^2 - 6\left(\frac{P}{kL}\right) + 1 = 0 \qquad \frac{P}{kL} = x$$

$$-4x^3 + 10x^2 - 6x + 1 = 0 \Rightarrow x =$$

$$-4\,x^3 + 10\,x^2 - 6\,x + 1 = 0 \xrightarrow[\textit{float}]{\textit{solve},x} \begin{bmatrix} 0.5 \\ 1.7071067811865475244 \\ 0.2928932188134524756 \end{bmatrix}$$

$$\left(\frac{P}{kL}\right)_1 = 0.2929 \Rightarrow P_{cr_1} = 0.2929\,kL$$

$$P_{cr_2} = 0.5\,kL$$

$$P_{cr_3} = 1.7071\,kL$$

Mathcad $\Rightarrow$

$$\det(A(k,p,l)) = 0 \xrightarrow[\substack{\textit{substitute},\,(k^2\cdot l^2)^{0.5} = k\cdot l \\ \textit{assume},\,k>0,\,l>0}]{\textit{solve},p} \begin{bmatrix} 0.5\cdot k\cdot l \\ 1.7071067811865475244\cdot k\cdot l \\ 0.2928932188134524756\cdot k\cdot l \end{bmatrix}$$

SHAH

$P_{cr_1} = 0.2929\,kL$,

$$\begin{bmatrix} k-\frac{2P}{L} & \frac{P}{L} & 0 \\ \frac{P}{L} & k-\frac{2P}{L} & \frac{P}{L} \\ 0 & \frac{P}{L} & k-\frac{2P}{L} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \bar{0} \Rightarrow \begin{bmatrix} \overbrace{k - 2\times 0.2929k}^{0.4142k} & 0.2929k & 0 \\ 0.2929k & 0.4142k & 0.2929k \\ 0 & 0.2929k & 0.4142k \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \bar{0}$$

$\delta_1 = 1 \Rightarrow \quad 0.4142k + 0.2929k\delta_2 = 0 \Rightarrow \delta_2 = -1.414$

$0.2929k\underbrace{\delta_2}_{-1.414} + 0.4142k\delta_3 = 0 \Rightarrow \delta_3 = 1$

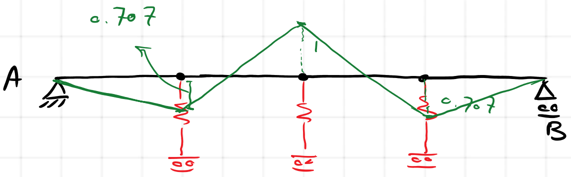
$$\delta = \begin{bmatrix} 1 \\ -1.414 \\ 1 \end{bmatrix} / 1.414$$

$$\rightarrow \quad \delta = \begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix}$$

$P_{cr_1} = 0.2929k$

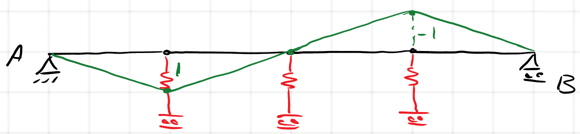
$$\delta = \begin{bmatrix} 0.707 \\ -1 \\ 0.707 \end{bmatrix}$$

Mode shape 1



$P_{cr_2} = 0.5\,kL$

$$A(k, 0.5\ k\cdot l, l) \rightarrow \begin{bmatrix} 0.0 & 0.5\cdot k & 0 \\ 0.5\cdot k & 0.0 & 0.5\cdot k \\ 0 & 0.5\cdot k & 0.0 \end{bmatrix}$$

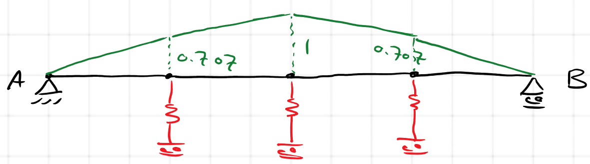


$\delta_1 = 1$, $0.5k\delta_2 = 0 \Rightarrow \delta_2 = 0$

$0.5k + 0.5k\delta_3 = 0 \Rightarrow \delta_3 = -1$

$P_{cr_3} = 1.7071kL$

$$\delta = \begin{bmatrix} 1 \\ 1.4142 \\ 1 \end{bmatrix} / 1.4142$$



$$\begin{bmatrix} -2.4142\cdot k & 1.7071\cdot k & 0 \\ 1.7071\cdot k & -2.4142\cdot k & 1.7071\cdot k \\ 0 & 1.7071\cdot k & -2.4142\cdot k \end{bmatrix}$$

$$\delta = \begin{bmatrix} 0.707 \\ 1 \\ 0.707 \end{bmatrix}$$

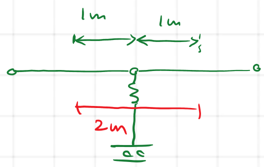
$-2.4142k + 1.7071k\delta_2 = 0 \Rightarrow \delta_2 = 1.4142$

$1.7071k(1.4142) - 2.4142k\delta_3 = 0 \Rightarrow \delta_3 = 1$

$C_u = 10 \text{ kPa}$

subgrade reaction coefficient : 100

subgrade reaction : $K_s = A \cdot \frac{C_u}{d_{eff}}$



$$K_{spring} = \underbrace{K_s}_{A \cdot \frac{Cu}{d_{eff}}} \cdot d_{eff} \cdot \underbrace{Spacing}_{2m}$$

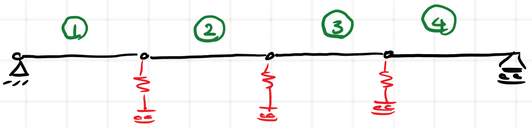
$$\Rightarrow K_{spring} = \underset{100}{A} \cdot C_u \cdot 2m = \underset{(m)}{200\, C_u}$$

$$C_u = 10 \text{ kPa} \rightarrow K_{spring} = \underset{(m)}{200} \times 10 \text{ kPa} = 2000 \text{ kN/m}$$

$$P_{cr_1} = 0.2929 \, K L = 0.2929 \times 2000 \text{ kN/m} \times 2m = 1171.6 \text{ kN}$$

$$P_{cr_2} = 2000 \text{ kN}$$

$$P_{cr_3} = 6828 \text{ kN}$$



$$L_{eff} = 2m$$

$$P_{cr} = \frac{\pi^2 E I}{L_{eff}^2} \qquad E = 210 \text{ GPa}, \; I = 1.402 \times 10^6 \text{ mm}^4$$

$$L_{eff} = 2m$$

Profile	Drawing	Profile dimensions		Area properties				Second moment of area
		Ext. Diameter D [mm]	Wall thickness t [mm]	Weight m [kg/m]	External perimeter P [m]	Area A [mm²]	Shear area A_v [mm²]	I [×10⁶ mm⁴]
CHS 88.9 / 6.3	dxf	88.9	6.3	12.8	0.279	1635	1041	1.402

$$P_{cr} = \frac{\pi^2 \times 210 \text{ GPa} \times 1.402 \times 10^6 \text{ mm}^4}{(2m)^2} = 726 \text{ kN}$$

First four modes will be $\boxed{726 \text{ kN}}$ 1171.6, 2000, 6828

Profile	Drawing	Profile dimensions		Area properties				Second moment of area
		Ext. Diameter D [mm]	Wall thickness t [mm]	Weight m [kg/m]	External perimeter P [m]	Area A [mm²]	Shear area A_v [mm²]	I [×10⁶ mm⁴]
CHS 168.3 / 10	dxf	168.3	10.0	39.0	0.529	4973	3166	15.64

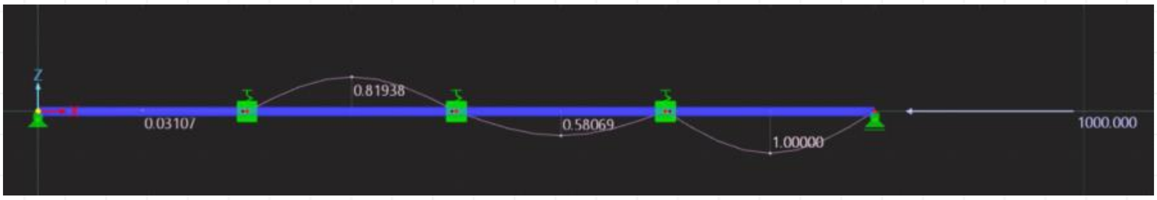
$\Rightarrow P_{cr} = 8104 \text{ kN}$ 1171.6, 2000, 6828, 8104

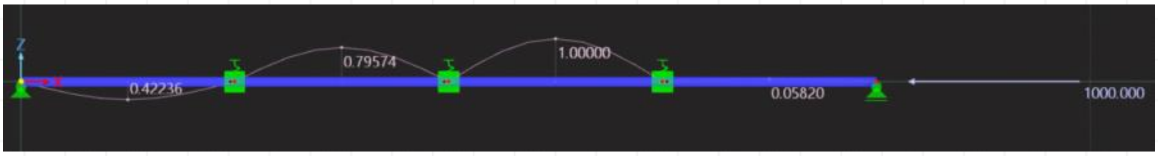
Profile	Drawing	Profile dimensions		Area properties				Second moment of area
		Ext. Diameter D [mm]	Wall thickness t [mm]	Weight m [kg/m]	External perimeter P [m]	Area A [mm²]	Shear area A_v [mm²]	I [×10⁶ mm⁴]
CHS 139.7 / 6.3	dxf	139.7	6.3	20.7	0.439	2640	1681	5.886

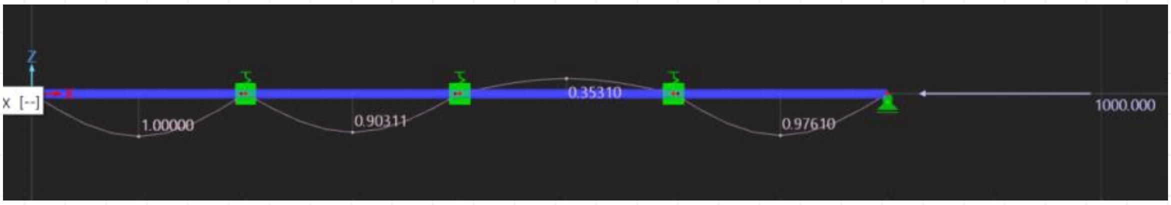
$\Rightarrow P_{cr} = 3050 \text{ kN}$

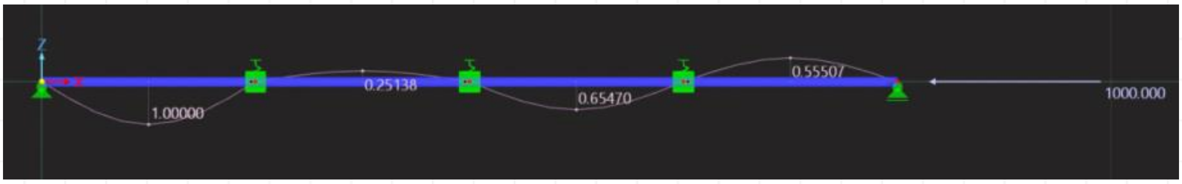
1171.6, 2000, 3050, 3050, 3050, 3050, 6828

The first four modes :



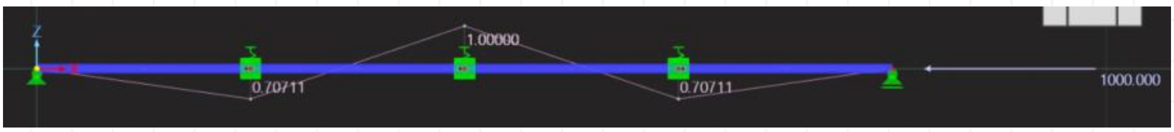






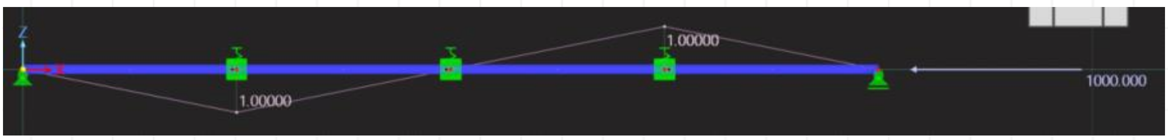
Mode No.	Critical Load Factor f [--]
1	0.718
2	0.718
3	0.718
4	0.718

$P_{cr} = 0.718 \times P_{applied} = 718 \text{ kN}$

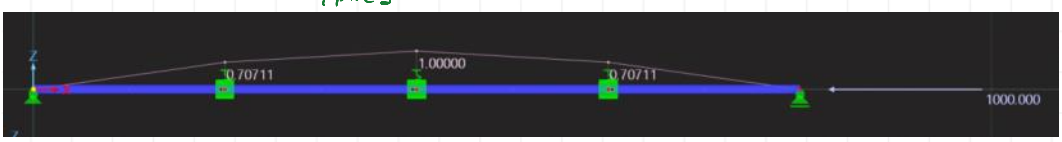


1.172
2.000

$P_{cr} = 1.172 \times P_{applied} = 1172 \text{ kN}$



$P_{cr} - 2 P_{applied} = 2000 \text{ kN}$



$P_{cr} = 6.828 P_{applied} = 6828 \text{ kN}$