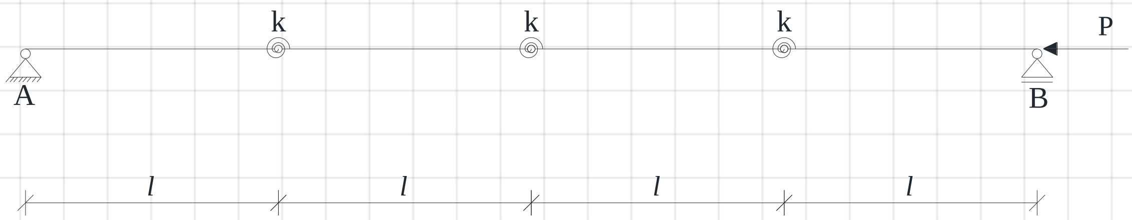
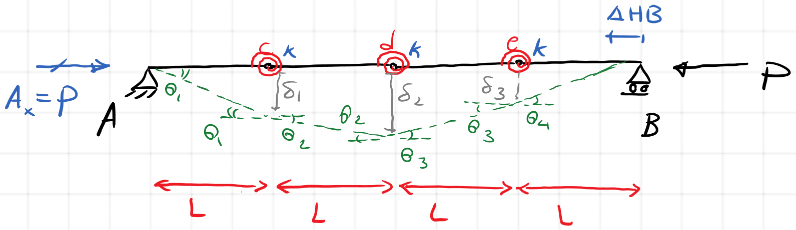
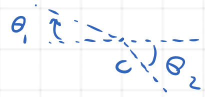
Four rigid bars form a compressive element under applied compressive load at point B. It is assumed that the bars are hinged and connected at their intersection. At one end, the element is supported by a hinge, and at the other end, point B, the element is supported by a roller. The system is assumed to be a 2D structure. At the intersections, there are three rotational springs with a spring coefficient of k. Each rigid bar is the length of l.

a) Determine the total potential energy of the system.
b) Determine the buckling loads.
c) Determine the buckling shape modes.
d) Assume the rotational spring coefficient is $2000 kN.m/rad$ and determine the buckling loads numerically.
e) Model the system in FEM software and compare your results.

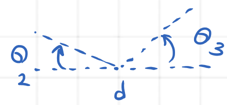






$|\theta| = \theta_1 - \theta_2$

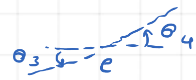
$$\Delta HB = L - L\cos\theta_1 + L - L\cos\theta_2 + L - L\cos\theta_3 + L - L\cos\theta_4$$



$$\Delta HB = L(1-\cos\theta_1 + 1-\cos\theta_2 + 1-\cos\theta_3 + 1-\cos\theta_4)$$

$|\theta| = \theta_2 + \theta_3$

$$\Delta HB = \frac{1}{L}(\delta_1^2 + \delta_2^2 + \delta_3^2 - \delta_1\delta_2 - \delta_2\delta_3)$$



$$-V = -P\cdot\Delta HB = -\frac{P}{L}(\delta_1^2 + \delta_2^2 + \delta_3^2 - \delta_1\delta_2 - \delta_2\delta_3)$$

$$W = \sum \frac{1}{2}k\cdot\theta^2 = \frac{1}{2}k\left((\theta_1-\theta_2)^2 + (\theta_2+\theta_3)^2 + (\theta_3-\theta_4)^2\right)$$

$|\theta| = \theta_3 - \theta_4$

$$\theta_1 = \frac{\delta_1}{L}, \quad \theta_2 = \frac{\delta_2-\delta_1}{L}, \quad \theta_3 = \frac{\delta_2-\delta_3}{L}, \quad \theta_4 = \frac{\delta_3}{L}$$

$$W = \frac{1}{2}k\left(\left(\frac{2\delta_1-\delta_2}{L}\right)^2 + \left(\frac{2\delta_2-\delta_1-\delta_3}{L}\right)^2 + \left(\frac{\delta_2-2\delta_3}{L}\right)^2\right)$$

$$\Pi(p,k,l,\delta_1,\delta_2,\delta_3) := W(k,l,\delta_1,\delta_2,\delta_3) + V(p,l,\delta_1,\delta_2,\delta_3)$$

$$\frac{\partial\pi}{\partial\delta_1} = 0, \quad \frac{\partial\pi}{\partial\delta_2} = 0, \quad \frac{\partial\pi}{\partial\delta_3} = 0 \Rightarrow$$

$$\frac{d}{d\delta_1}\Pi(p,k,l,\delta_1,\delta_2,\delta_3) \xrightarrow{expand} \frac{\delta_2\cdot p}{l} - \frac{2\cdot\delta_1\cdot p}{l} + \frac{\delta_3\cdot k}{l^2} - \frac{4\cdot\delta_2\cdot k}{l^2} + \frac{5\cdot\delta_1\cdot k}{l^2}$$

$$\frac{d}{d\delta_2}\Pi(p,k,l,\delta_1,\delta_2,\delta_3) \xrightarrow{expand} \frac{\delta_3\cdot p}{l} - \frac{2\cdot\delta_2\cdot p}{l} + \frac{\delta_1\cdot p}{l} - \frac{4\cdot\delta_3\cdot k}{l^2} + \frac{6\cdot\delta_2\cdot k}{l^2} - \frac{4\cdot\delta_1\cdot k}{l^2}$$

$$\frac{d}{d\delta_3}\Pi(p,k,l,\delta_1,\delta_2,\delta_3) \xrightarrow{expand} -\frac{2\cdot\delta_3\cdot p}{l} + \frac{\delta_2\cdot p}{l} + \frac{5\cdot\delta_3\cdot k}{l^2} - \frac{4\cdot\delta_2\cdot k}{l^2} + \frac{\delta_1\cdot k}{l^2}$$

$$\frac{\delta_2 \cdot p}{l} - \frac{2\cdot\delta_1 \cdot p}{l} + \frac{\delta_3 \cdot k}{l^2} - \frac{4\cdot\delta_2 \cdot k}{l^2} + \frac{5\cdot\delta_1 \cdot k}{l^2}$$

$$\frac{\delta_3 \cdot p}{l} - \frac{2\cdot\delta_2 \cdot p}{l} + \frac{\delta_1 \cdot p}{l} - \frac{4\cdot\delta_3 \cdot k}{l^2} + \frac{6\cdot\delta_2 \cdot k}{l^2} - \frac{4\cdot\delta_1 \cdot k}{l^2}$$

$$-\frac{2\cdot\delta_3 \cdot p}{l} + \frac{\delta_2 \cdot p}{l} + \frac{5\cdot\delta_3 \cdot k}{l^2} - \frac{4\cdot\delta_2 \cdot k}{l^2} + \frac{\delta_1 \cdot k}{l^2}$$

$$\begin{cases} \left(\frac{5k}{L^2} - \frac{2P}{L}\right)\delta_1 + \left(-\frac{4k}{L^2} + \frac{P}{L}\right)\delta_2 + \frac{k}{L^2}\delta_3 = 0 \\ \left(-\frac{4k}{L^2} + \frac{P}{L}\right)\delta_1 + \left(\frac{6k}{L^2} - \frac{2P}{L}\right)\delta_2 + \left(-\frac{4k}{L^2} + \frac{P}{L}\right)\delta_3 = 0 \\ \frac{k}{L^2}\delta_1 + \left(-\frac{4k}{L^2} + \frac{P}{L}\right)\delta_2 + \left(\frac{5k}{L^2} - \frac{2P}{L}\right)\delta_3 = 0 \end{cases}$$

$$A = \begin{bmatrix} \frac{5k}{L^2} - \frac{2P}{L} & -\frac{4k}{L^2} + \frac{P}{L} & \frac{k}{L^2} \\ -\frac{4k}{L^2} + \frac{P}{L} & \frac{6k}{L^2} - \frac{2P}{L} & -\frac{4k}{L^2} + \frac{P}{L} \\ \frac{k}{L^2} & -\frac{4k}{L^2} + \frac{P}{L} & \frac{5k}{L^2} - \frac{2P}{L} \end{bmatrix} \qquad X = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix}, \quad A \cdot X = \bar{0}$$

$\det(A) = 0 \implies$

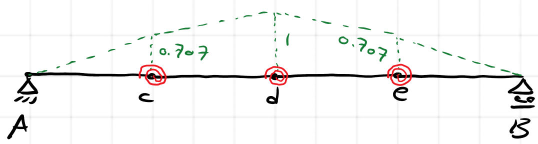
$$\det(A(p,k,l)) = 0 \xrightarrow[\text{assume}, k>0, l>0]{\text{solve}, p \quad \text{substitute}, (k^2 \cdot l^2)^{0.5} = k\cdot l} \begin{bmatrix} \frac{2.0\cdot k}{l} \\ \frac{3.4142135623730950488\cdot k}{l} \\ \frac{0.5857864376269049512\cdot k}{l} \end{bmatrix}$$

$(P_{cr})_1 = 0.5858\frac{k}{L}$ $\qquad (P_{cr})_2 = 2\frac{k}{L}$ $\qquad (P_{cr})_3 = 3.4142\frac{k}{L}$

$(P_{cr})_1 = 0.5858\frac{k}{L}$

$$\begin{bmatrix} \frac{3.8284271247461900976\cdot k}{l^2} & \frac{-3.4142135623730950488\cdot k}{l^2} & \frac{k}{l^2} \\ \frac{-3.4142135623730950488\cdot k}{l^2} & \frac{4.8284271247461900976\cdot k}{l^2} & \frac{-3.4142135623730950488\cdot k}{l^2} \\ \frac{k}{l^2} & \frac{-3.4142135623730950488\cdot k}{l^2} & \frac{3.8284271247461900976\cdot k}{l^2} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \bar{0}$$

$$3.8284\cancel{\frac{k}{L^2}}\delta_1 - 3.4142\cancel{\frac{k}{L^2}}\delta_2 + \cancel{\frac{k}{L^2}}\delta_3 = 0$$



$\delta_1 = 1 \Rightarrow -3.4142\delta_2 + \delta_3 = -3.8284$

$-3.4142\delta_2 + 3.8284\delta_3 = -1$

$\delta_2 = 1.414, \ \delta_3 = 1$

$$\begin{bmatrix} 1 \\ 1.414 \\ 1 \end{bmatrix} / 1.414 \Rightarrow \begin{bmatrix} 0.707 \\ 1 \\ 0.707 \end{bmatrix}$$

$$\frac{A\left(D(k,l)_0,k,l\right)\cdot l^2}{k} \rightarrow \begin{bmatrix}1.0 & -2.0 & 1.0\\ -2.0 & 2.0 & -2.0\\ 1.0 & -2.0 & 1.0\end{bmatrix}$$

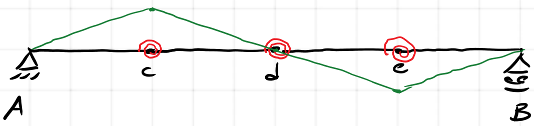
$\delta_1 = 1 \Rightarrow \quad \delta_1 - 2\delta_2 + \delta_3 = 0$

$$\begin{cases} -2\delta_2 + \delta_3 = -1 \\ +2\delta_2 - 2\delta_3 = 2 \end{cases}$$

$\delta_2 = 0$

$\delta_3 = -1$

$$\begin{bmatrix}1\\0\\-1\end{bmatrix}$$



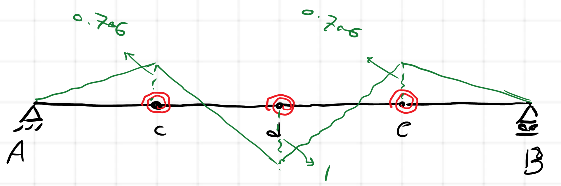
$$\frac{A\left(D(k,l)_1,k,l\right)\cdot l^2}{k} \xrightarrow{float,3} \begin{bmatrix}-1.83 & -0.586 & 1.0\\ -0.586 & -0.828 & -0.586\\ 1.0 & -0.586 & -1.83\end{bmatrix}$$

$-1.838\delta_1 - 0.586\delta_2 + \delta_3 = 0$

$-0.586\delta_1 - 0.828\delta_2 - 0.586\delta_3 = 0$

$\delta_1 = 1 \Rightarrow \begin{cases}\delta_2 = -1.416\\ \delta_3 = 1\end{cases}$ $\quad \begin{bmatrix}1\\-1.416\\1\end{bmatrix} / 1.416$

$$\begin{bmatrix}0.706\\-1\\0.706\end{bmatrix}$$



$P_{applied} = 1000\ kN$

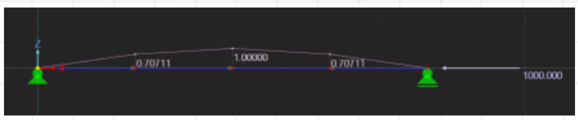
Mode No.	Critical Load Factor f [--]
1	0.584
2	1.993
3	3.404

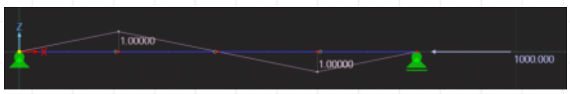
$K = 2000\ \frac{kN\cdot m}{rad}$, $L = 2m$

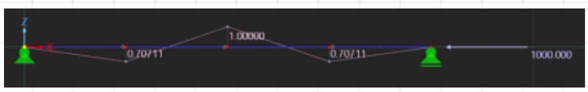
$P_{cr1} = 0.5858\ \frac{K}{L} = 0.5858 \times \frac{2000\ \frac{kN\cdot m}{rad}}{2\ m} = 585.8\ kN$

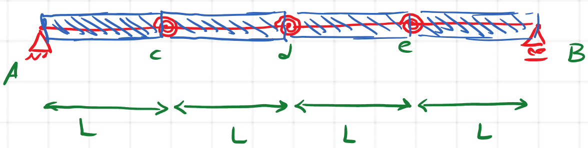
$P_{cr2} = 2\ \frac{K}{L} = 2000\ kN$

$P_{cr3} = 3.4142\ \frac{K}{L} = 3414\ kN$









Profile	Drawing	Profile dimensions		Area properties				
		Ext. Diameter D [mm]	Wall thickness t [mm]	Weight m [kg/m]	External perimeter P [m]	Area A [mm^2]	Shear area A_v [mm^2]	Second moment of area I [×10^6 mm^4]
CHS 88.9 / 6.3	dxf	88.9	6.3	12.8	0.279	1635	1041	1.402

$L_{eff} = 2m$

$$P_{cr} = \frac{\pi^2 E I}{L^2} = 726 \text{ kN}$$

Profile	Drawing	Profile dimensions		Area properties				
		Ext. Diameter D [mm]	Wall thickness t [mm]	Weight m [kg/m]	External perimeter P [m]	Area A [mm^2]	Shear area A_v [mm^2]	Second moment of area I [×10^6 mm^4]
CHS 139.7 / 6.3	dxf	139.7	6.3	20.7	0.439	2640	1681	5.886

$P_{cr} = 3050 \text{ kN}$

Profile	Drawing	Profile dimensions		Area properties				
		Ext. Diameter D [mm]	Wall thickness t [mm]	Weight m [kg/m]	External perimeter P [m]	Area A [mm^2]	Shear area A_v [mm^2]	Second moment of area I [×10^6 mm^4]
CHS 168.3 / 10	dxf	168.3	10.0	39.0	0.529	4973	3166	15.64

$P_{cr} = 8104 \text{ kN}$