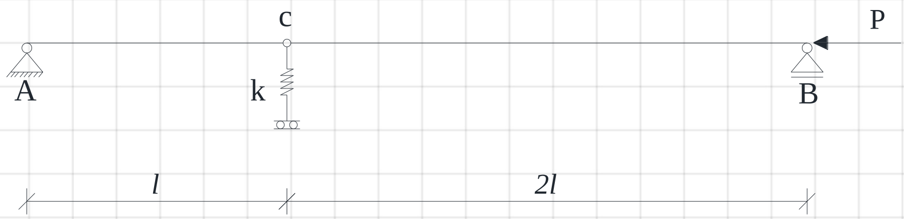
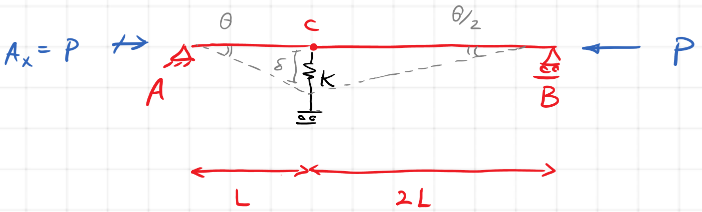
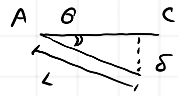
Two rigid elements are connected with a hinge at point *c* with a translational spring with a constant stiffness of *k*. The system is under a compressive load of P at the end B, supported by a roller. By increasing the compressive load, the system will buckle.

a) Determine the total potential energy of the system.
b) Determine the primary and secondary paths.
c) Check the paths in terms of being stable or unstable.







$$\sin\theta = \frac{\delta}{L}$$

$$\rightarrow \delta = L\,\sin\theta$$

$$\Delta HB = L - L\cos\theta + 2L - 2L\cos\frac{\theta}{2}$$

$$\Delta HB = L\left(1 - \cos\theta + 2 - 2\cos\frac{\theta}{2}\right)$$

$$V = -P\cdot\Delta HB = -PL\left(1 - \cos\theta + 2 - 2\cos\frac{\theta}{2}\right)$$

$$W = \frac{1}{2}k\cdot\delta^2 = \frac{1}{2}\cdot k\cdot(L\sin\theta)^2 = \frac{kL^2}{2}\sin^2\theta$$

$$\pi(\theta) = W + V = \frac{kL^2}{2}\sin^2\theta - PL\left(1 - \cos\theta + 2 - 2\cos\frac{\theta}{2}\right)$$

1) $\left.\begin{array}{l}\sin\theta \simeq \theta \\ 1-\cos\theta \simeq \frac{1}{2}\theta^2\end{array}\right\} \Rightarrow \pi(\theta) = \frac{kL^2\theta^2}{2} - PL\left(\frac{1}{2}\theta^2 + \left(\frac{\theta}{2}\right)^2\right)$

$$\pi(\theta) = \frac{kL^2\theta^2}{2} - PL\left(\frac{3}{4}\theta^2\right)$$

$$\frac{\partial\pi}{\partial\theta} = kL^2\theta - \frac{3}{2}PL\theta = 0 \Longrightarrow \begin{cases}\theta = 0 \\ P = \frac{2}{3}k\cdot L = P_0\end{cases}$$

2) $\pi(\theta) = \frac{kL^2}{2}\sin^2\theta - PL\left(1 - \cos\theta + 2 - 2\cos\frac{\theta}{2}\right)$

$$\frac{\partial\pi(\theta)}{\partial\theta} = \frac{kL^2}{\cancel{2}}\cdot\cancel{2}\sin\theta\cos\theta - PL\left(\sin\theta + \sin\frac{\theta}{2}\right) = 0$$

$$kL^{\cancel{2}}\cdot\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\cos\theta\right) - P\cancel{L}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) = 0$$

$$\sin\frac{\theta}{2}\left[kL\left(2\cos\frac{\theta}{2}\cos\theta\right) - P\left(2\cos\frac{\theta}{2} + 1\right)\right] = 0$$

Secondary Path

$\frac{P}{P_0}$

$\rightarrow \sin\frac{\theta}{2} = 0 \Rightarrow \theta = 0 \Rightarrow$ Primary path

$$\rightarrow kL\left(2\cos\frac{\theta}{2}\cos\theta\right) - P\left(2\cos\frac{\theta}{2} + 1\right) = 0 \Rightarrow \frac{P}{\frac{2}{3}kL} = \frac{kL}{\frac{2}{3}kL}\cdot\frac{2\cos\frac{\theta}{2}\cos\theta}{1 + 2\cos\frac{\theta}{2}} \Rightarrow \lambda = 3\,\frac{\cos\frac{\theta}{2}\cos\theta}{1 + 2\cos\frac{\theta}{2}}$$

$$\frac{\partial \pi(\theta)}{\partial \theta} = \frac{kL^2}{\cancel{2}} \cdot \cancel{2}\sin\theta\cos\theta - PL\left(\sin\theta + \sin\frac{\theta}{2}\right) = 0$$

$$\frac{\partial^2 \pi}{\partial \theta^2} = kL^2\left(\cos\theta\cos\theta - \sin\theta\sin\theta\right) - PL\left(\cos\theta + \frac{1}{2}\cos\frac{\theta}{2}\right)$$

$$kL^2\cos 2\theta - PL\left(\cos\theta + \frac{1}{2}\cos\frac{\theta}{2}\right) > 0 \quad \text{(stable)}$$

P_0

Primary path $\theta = 0 \Rightarrow kL^2 - PL\left(1 + \frac{1}{2}\right) > 0 \Rightarrow kL^2 > PL \cdot \frac{3}{2} \Rightarrow P < \frac{2}{3}kL$ stable

$$\frac{P}{P_0} = \lambda < 1$$

secondary path:

$$kL^2\cos 2\theta - PL\left(\cos\theta + \frac{1}{2}\cos\frac{\theta}{2}\right) > 0 \Rightarrow \frac{P}{\frac{2}{3}kL} < \frac{kL}{\frac{2}{3}kL} \cdot \frac{\cos 2\theta}{\cos\theta + \frac{1}{2}\cos\frac{\theta}{2}}$$

$$\lambda < \frac{3}{2} \cdot \frac{\cos 2\theta}{\cos\theta + \frac{1}{2}\cos\frac{\theta}{2}}$$