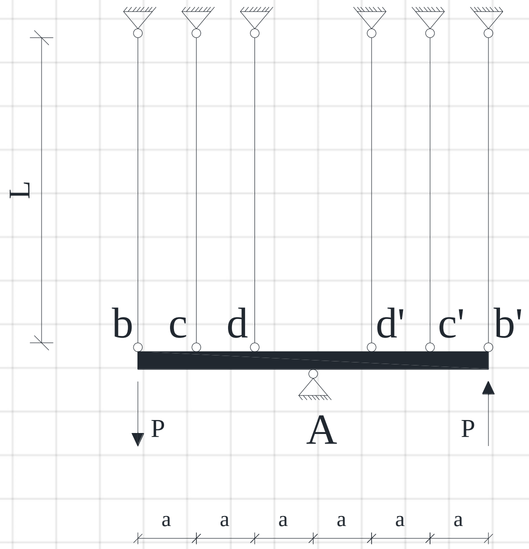
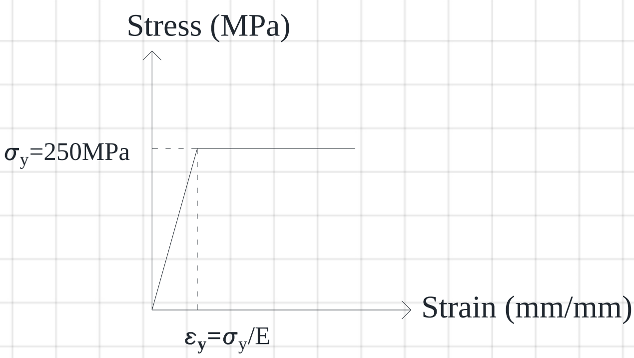
This video teaches us how to determine the elastic and plastic load for a rigid element connected to deformable bars. The example is an introduction to understanding how to determine the elastic and plastic bending moment of an Euler-Bernoulli cross-section. The example description is as follows:

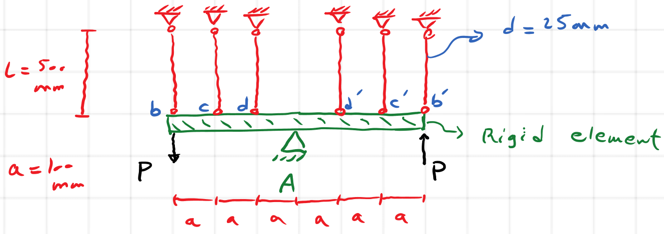
A rigid horizontal element with a length of *6a* is supported by a hinge at its mid-span. A couple of P forces are expected to be applied at both ends of the rigid component. With the interval of *a* from both ends, the element is supported by 6 rods. Rods are made from an elastic, perfectly plastic material. The yield limit of the material is 250MPa, and the elasticity modulus is 200GPa. The rods are with a diameter of 25mm with a length of 500mm. Assume the material is homogenous and has similar behavior in tension and compression. Also, we can ignore the effect of buckling in this example.

a) Derive the compatibility equations.
b) Determine the minimum required force P that the first bar(s) would yield.
c) Determine the load P when the next bar(s) would yield.
d) Determine the load P when the entire system becomes plastic[1].
e) Sketch the Force P and its node displacement graph.





[1] This is the moment that the structure becomes a mechanism.

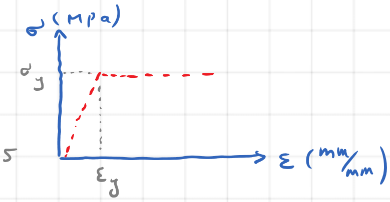


Elastic - perfectly plastic

$\sigma_y = 250\ MPa$

$E = 200\ GPa$

$\varepsilon_y = \dfrac{\sigma_y}{E} = 0.00125$



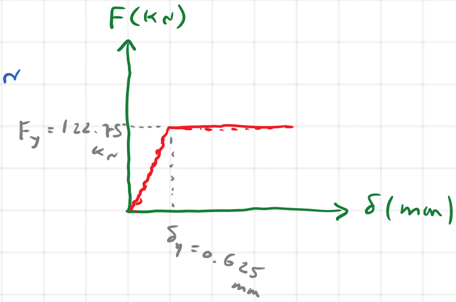
$\sigma_y = 250\ MPa$

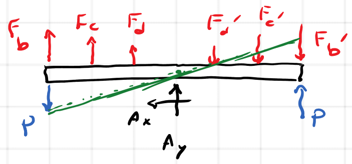
$A = \pi \times \dfrac{d^2}{4} = \dfrac{\pi (25\ mm)^2}{4} = 491\ mm^2$

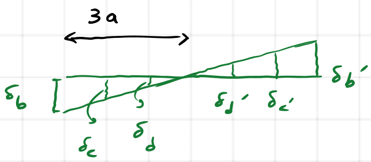
$\sigma = \dfrac{F}{A} \Rightarrow F_y = \sigma_y \cdot A = 250 MPa \times 491\ mm^2 = 122.75\ kN$

$\delta_Y = \dfrac{F_y \cdot L}{A\ E} = \dfrac{122.75\ kN \times 500\ mm}{491\ mm^2 \times 200\ GPa} = 0.625\ mm$

$\varepsilon = \dfrac{\delta}{L} \rightarrow \delta_Y = L \cdot \varepsilon_y = 500\ mm \times 0.00125 = 0.625\ mm$





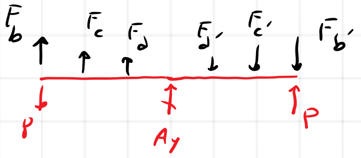


$\dfrac{\delta_b}{3\cancel{a}} = \dfrac{\delta_c}{2\cancel{a}} = \dfrac{\delta_d}{\cancel{a}}$ $\qquad$ $\dfrac{\delta_b'}{3\cancel{a}} = \dfrac{\delta_c'}{2\cancel{a}} = \dfrac{\delta_d'}{\cancel{a}}$

$\boxed{\begin{cases} \delta_b = 3\delta_d \\ \delta_c = 2\delta_d \end{cases}}$ (set I) $\boxed{\begin{cases} \delta_b' = 3\delta_d' \\ \delta_c' = 2\delta_d' \end{cases}}$ $\begin{cases} \delta_b = \delta_b' \\ \delta_c = \delta_c' \\ \delta_d = \delta_d' \end{cases}$

all bars in elastic phase: → Elastic equation is valid.

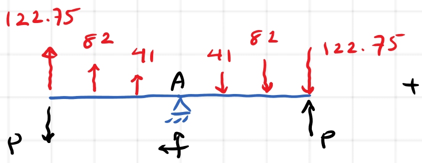
$\delta = \dfrac{F \cdot L}{A \cdot E}$, $\delta_b = \dfrac{F_b \cdot L}{A \cdot E}$, $\delta_c = \dfrac{F_c \cdot L}{A \cdot E}$, $\delta_d = \dfrac{F_d \cdot L}{A\ E}$



Assume $F_b = F_{b'} = F_y = 122.75\,kN$

(set I) $\rightarrow$ $\delta_b = 3\,\delta_d$, $\dfrac{F_b \cdot \cancel{L}}{\cancel{A \cdot E}} = 3\,\dfrac{F_d \cdot \cancel{L}}{\cancel{A E}} \rightarrow F_d = \frac{1}{3} F_b \rightarrow F_d = \frac{1}{3} \times 122.75\,kN \cong 41\,kN < F_y = 122.75\,kN$

$\delta_c = 2\,\delta_d$, $\dfrac{F_c \cdot \cancel{L}}{\cancel{AE}} = 2\,\dfrac{F_d \cdot \cancel{L}}{\cancel{AE}} \rightarrow F_c = 2F_d \rightarrow F_c \cong 82\,kN < F_y = 122.75\,kN$



$+\circlearrowleft \Sigma M_A = 0$

$P\cdot(3a) + P\cdot(3a) - 122.75\times(3a)\times 2 - 82\times(2a)\times 2 - 41\times(a)\times 2 = 0$

$\rightarrow P_{1} = 191\,kN$

$F_b = F_y = 122.75\,kN \rightarrow \delta_b = \delta_y = 0.625\,mm$

$\sigma_b = \sigma_y = 250\,MPa \rightarrow \varepsilon_b = \varepsilon_y = 0.00125$

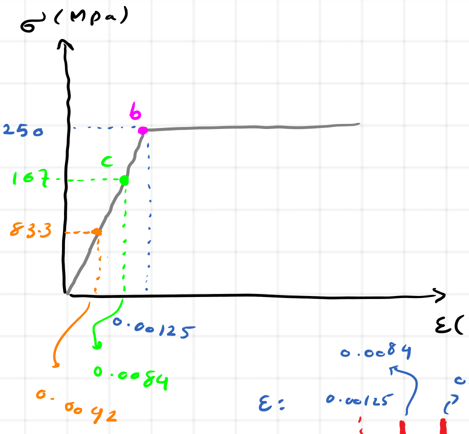
(c) , $F_c = 82\,kN \rightarrow \delta_c = \dfrac{F \cdot L}{A\,E} = \dfrac{82\,kN \times 500\,mm}{491\,mm^2 \times 200\,GPa} = 0.42\,mm$,

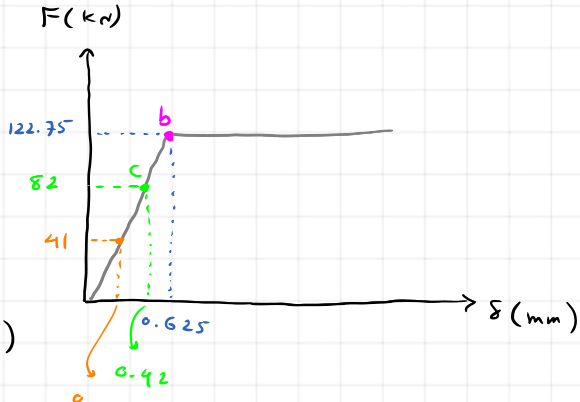
$\rightarrow \sigma = \dfrac{F}{A} = \dfrac{82000\,N}{491\,mm^2} = 167\,MPa$

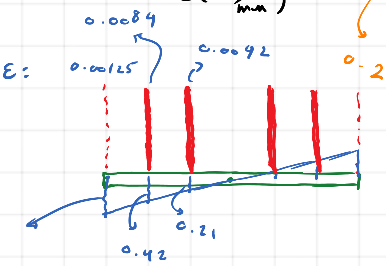
$\varepsilon = \dfrac{\sigma}{E} = \dfrac{167\,MPa}{200\,GPa} = 0.00084$

d), $F_d = 41\,kN$, $\delta_d = 0.21\,mm$, $\sigma = 83.3\,MPa$, $\varepsilon = 0.00042$

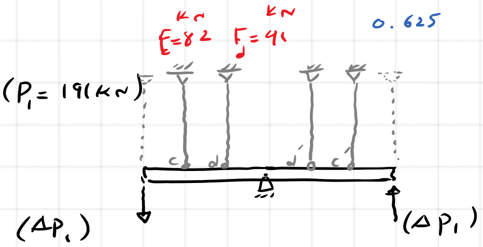
(−) (b)
(−) (c)
(−) (d)





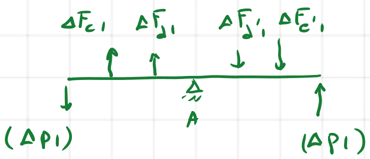


$P_{①} = 191 \text{ kN}$



capacity left for element c → $122.75^{kN} - 82^{kN} = 40.75 \text{ kN}$

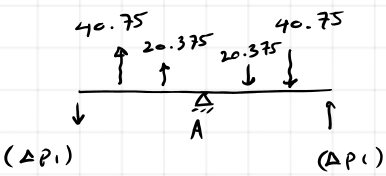
" " " " d → $122.75 - 41 \text{ kN} = 81.75 \text{ kN}$



Assume c & c′ become plastic before d & d′.

$\Delta F_{c1} = 40.75 \text{ kN}$

(Set I) → $\delta_c = 2\delta_d$ → $\frac{F_c \cdot \cancel{L}}{\cancel{AE}} = \frac{2 F_d \cdot \cancel{L}}{\cancel{AE}}$ → $(\Delta F_{d1}) = \frac{(\Delta F_{c1})}{2} = 20.375 \text{ kN} < 81.75 \text{ kN}$ (OK)



$\overset{+}{\circlearrowleft} \Sigma M_A = 0$

$(\Delta P_1) \cdot 3a \cdot 2 - 40.75 \times 2a \times 2 - 20.375 \times a \times 2 = 0 \Rightarrow (\Delta P_1) \cong 34 \text{ kN}$

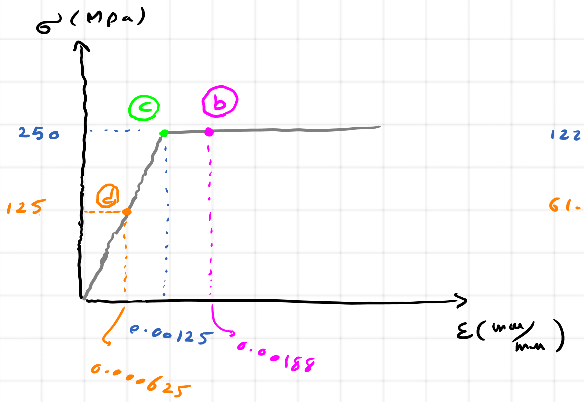
$F_c = (F_c) + (\Delta F_{c1}) = 122.75 \text{ kN}$ → $\delta_c = 0.625 \text{ mm}$

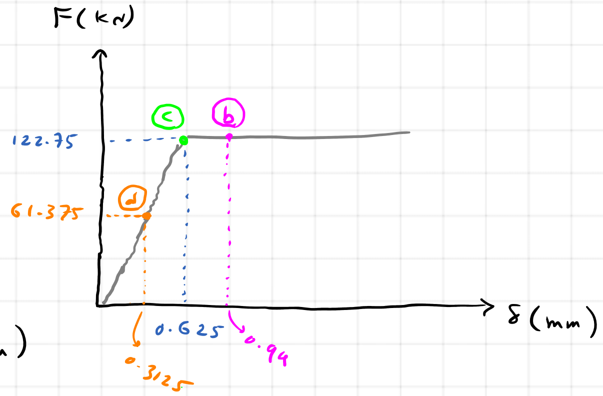
$F_d = (F_d) + (\Delta F_{d1}) = 41 + 20.375 = 61.375 \text{ kN}$ → $\delta_d = \frac{F_d \cdot L}{AE} = \frac{61.375 \text{ kN} \times 500^{mm}}{491 \text{ mm}^2 \times 200 \text{ GPa}} = 0.3125 \text{ mm}$

(b) → $F_b = 122.75 \text{ kN}$, $\delta_b = 3\delta_d = 0.94 \text{ mm} > \delta_y = 0.625 \text{ mm}$

Not valid $\left(\varepsilon \neq \frac{\sigma}{E}\right)$ $\varepsilon = \frac{\delta}{L} = \frac{0.94 \text{ mm}}{500 \text{ mm}} = 0.00188 > \varepsilon_y = 0.00125$

(–) (b)
(–) (c)
(–) (d)



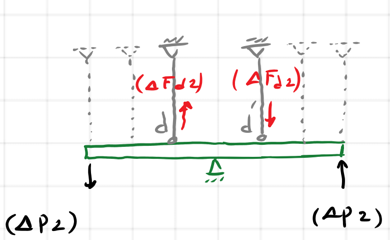


$$P_2 = P_1 + \Delta P_1 = 191\,kN + 34\,kN = 225\,kN$$

$P = P_{①} = 191\ kN\ ,\quad \delta_b = 0.625\,mm$

$P = P_{②} = 225\,kN\ ,\quad \delta_b = 0.94\,mm$

$F_d = 61.375\ kN$



$(\Delta F_{d2}) = 122.75\,kN - 61.375\,kN = 61.375\,kN$

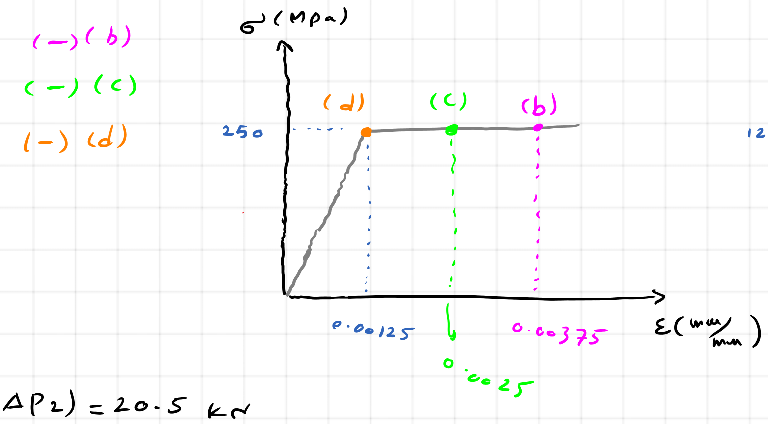
$\overset{+}{\circlearrowleft} \Sigma M_A = 0 \rightarrow (\Delta P_2)\cdot 3a\cdot 2 - \underbrace{(\Delta F_d 2)}_{61.375\,kN}\cdot a\times 2 = 0 \quad \rightarrow (\Delta P_2) \cong 20.5\,kN$

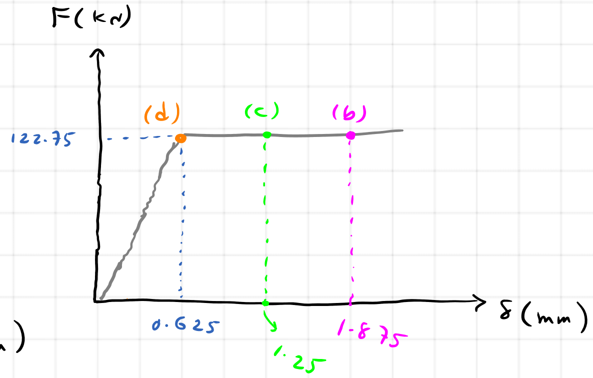
$$F_d = \underbrace{61.375}_{F_d + \Delta F_{d1}} + \underbrace{61.375}_{(\Delta F_d 2)} = 122.75\,kN \longrightarrow \delta_d = 0.625,\quad \varepsilon_d = 0.00125,\quad \sigma_d = \sigma_y = 250\,Mpa$$

c & b $\rightarrow$ $\begin{cases} F_b = 122.75\,kN \\ F_c = 122.75\,kN \end{cases}$

$\delta_b = 3\delta_d = 3\times 0.625\,mm = 1.875\,mm,\quad \varepsilon = \frac{\delta}{L} = 0.00375 > 0.00125$

$\delta_c = 2\delta_d = 2\times 0.625\,mm = 1.25\,mm,\quad \varepsilon = \frac{\delta}{L} = 0.0025 \geq 0.00125$

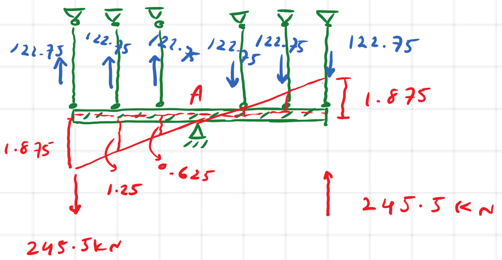




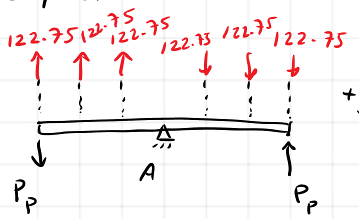
$(\Delta P_2) = 20.5\ kN$

$P_3 = P_2 + \Delta P_2 = 225\,kN + 20.5\,kN = 245.5\,kN$

$\delta_b = 1.875\ mm$

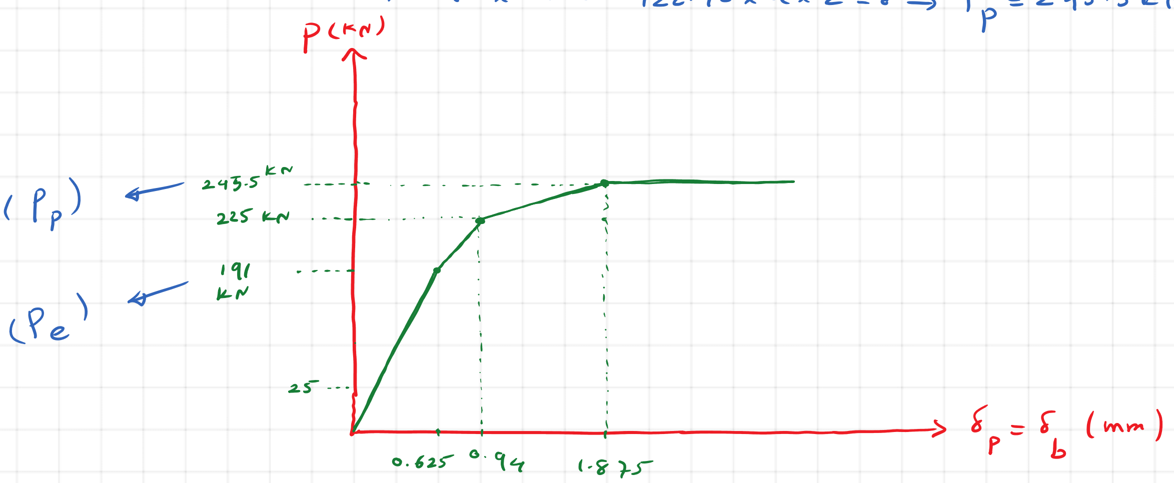


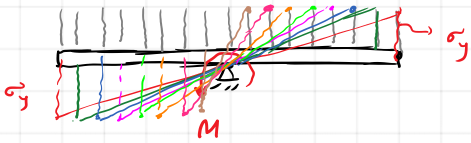
Determine the plastic load of the shown system:



$\circlearrowleft^{+} \sum M_A = 0$

$P_p \times 3a \times 2 - 122.75 \times 3a \times 2 - 122.75 \times 2a \times 2 - 122.75 \times a \times 2 = 0 \rightarrow P_p = 245.5\,kN$




σ_y
σ_y
M