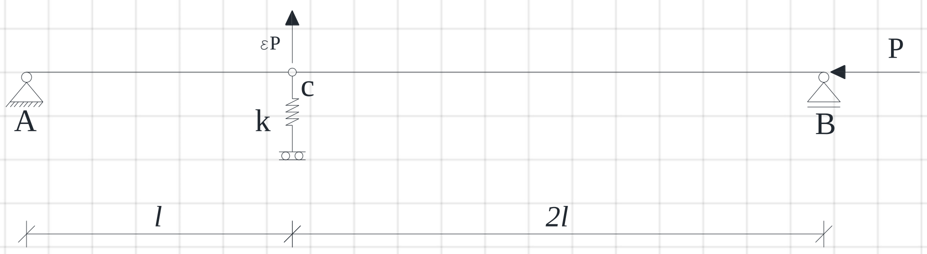
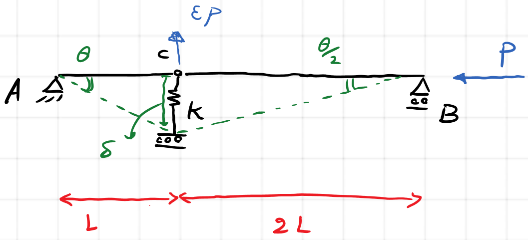
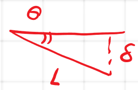
Two rigid elements are connected with a hinge at point c with a translational spring with a constant stiffness of k. The system is under a compressive load of P at the end B, supported by a roller. The system can be under a perturbation load like εP or imperfection in the erection phase with an initial angle of θ_0. In both conditions:

a) Determine the total potential energy of the system.
b) Sketch the equilibrium buckling path with different values of e or θ_0.
c) Determine the bifurcation point.







$$\sin\theta = \frac{\delta}{L}$$

$$\delta = L\cdot\sin\theta$$

$$\Delta_{HB} = L - L\cos\theta + 2L - 2L\cos(\theta/2)$$

$$V = -P\cdot\Delta_{HB} = -PL\left(1-\cos\theta + 2\left(1-\cos\frac{\theta}{2}\right)\right) - \varepsilon P\cdot\delta$$

$$V = -PL\left(1-\cos\theta + 2\left(1-\cos\frac{\theta}{2}\right)\right) - \varepsilon P\cdot L\cdot\sin\theta$$

$$W = \frac{1}{2}k\cdot\delta^2 = \frac{1}{2}k\,(L\sin\theta)^2 = \frac{kL^2}{2}\sin^2\theta$$

$$\pi(\theta) = W + V = \frac{kL^2}{2}\cdot\sin^2\theta - PL\left(1-\cos\theta + 2\left(1-\cos\frac{\theta}{2}\right)\right) - \varepsilon\cdot P\cdot L\cdot\sin\theta$$

1)

$$\left.\begin{array}{l}\sin\theta \simeq \theta \\ \theta\to 0 \\ 1-\cos\theta \simeq \frac{1}{2}\theta^2 \\ \theta \to 0\end{array}\right\} \Rightarrow \pi(\theta) = \frac{kL^2}{2}\theta^2 - PL\left(\frac{1}{2}\theta^2 + 2\cdot\frac{1}{2}\left(\frac{\theta}{2}\right)^2\right) - \varepsilon P L\theta$$

$$\pi(\theta) = \frac{kL^2}{2}\theta^2 - \frac{3}{4}PL\theta^2 - \varepsilon PL\theta$$

$$\frac{\partial\pi}{\partial\theta} = kL^2\theta - \frac{3}{2}PL\theta - \varepsilon PL = 0 \Rightarrow PL\left(\frac{3}{2}\theta + \varepsilon\right) = kL^2\theta$$

$$\left.\begin{array}{l} P = \dfrac{kL}{\frac{3}{2}\theta+\varepsilon}\cdot\theta \\ \varepsilon = 0\end{array}\right\} \Rightarrow \boxed{P_0 = \frac{2}{3}kL}$$

2)

$$\pi(\theta) = \frac{kL^2}{2}\sin^2\theta - PL\left(1-\cos\theta + 2\left(1-\cos\frac{\theta}{2}\right)\right) - \varepsilon P\cdot L\sin\theta$$

$$\frac{\partial\pi}{\partial\theta} = kL^2\sin\theta\cos\theta - PL\left(\sin\theta + \sin\frac{\theta}{2}\right) - \varepsilon P\cdot L\cdot\cos\theta = 0$$

$$PL\left[\sin\theta + \sin\frac{\theta}{2} + \varepsilon\cos\theta\right] = kL^2\sin\theta\cos\theta$$

$$\frac{P}{\frac{2}{3}kL} = \frac{kL}{\frac{2}{3}kL}\,\frac{\sin\theta\cos\theta}{\sin\theta + \sin\frac{\theta}{2} + \varepsilon\cos\theta} \Rightarrow \frac{P}{P_0} = \lambda = \frac{3}{2}\cdot\frac{\sin\theta\cos\theta}{\sin\theta + \sin\frac{\theta}{2} + \varepsilon\cos\theta}$$

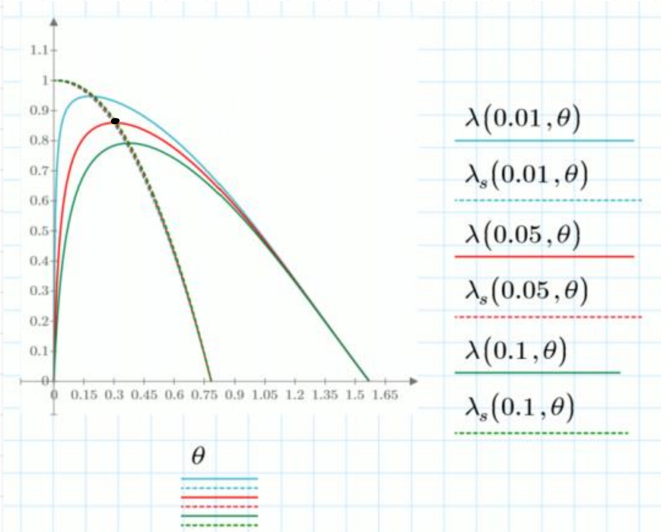
$$\frac{\partial \pi}{\partial \theta} = kL^2 \sin\theta \cos\theta - PL\left(\sin\theta + \sin\frac{\theta}{2}\right) - \varepsilon P \cdot L \cdot \cos\theta = 0$$

$$\frac{\partial^2 \pi}{\partial \theta^2} = kL^2(\cos\theta\cos\theta - \sin\theta\sin\theta) - PL\left(\cos\theta + \frac{1}{2}\cos\frac{\theta}{2}\right) + \varepsilon PL\sin\theta > 0$$

$$\lambda \left(\frac{PL}{\frac{2}{3}kL}\right)\left(\cos\theta + \frac{1}{2}\cos\frac{\theta}{2} - \varepsilon\sin\theta\right) < \frac{kL^2}{\frac{2}{3}kL}(\cos 2\theta)$$

$$\lambda < \frac{3}{2} \cdot \frac{\cos 2\theta}{\cos\theta + \frac{1}{2}\cos\frac{\theta}{2} - \varepsilon\sin\theta}$$

$$\lambda = \frac{3}{2} \cdot \frac{\sin\theta\cos\theta}{\sin\theta + \sin\frac{\theta}{2} + \varepsilon\cos\theta}$$



Assume → Define bifurcation point:

$\varepsilon = 0.05$

$$\cancel{\frac{3}{2}}\frac{\sin\theta\cos\theta}{\sin\theta + \sin\frac{\theta}{2} + 0.05\cos\theta} = \cancel{\frac{3}{2}}\frac{\cos 2\theta}{\cos\theta + \frac{1}{2}\cos\frac{\theta}{2} - 0.05\sin\theta}\Bigg|_{\theta_{guess} = 0.3\ \text{rad}}$$

Guess Values

$\theta_g := 0.3$

Constraints

$\lambda(\varepsilon, \theta_g) = \lambda_s(\varepsilon, \theta_g)$

Solver

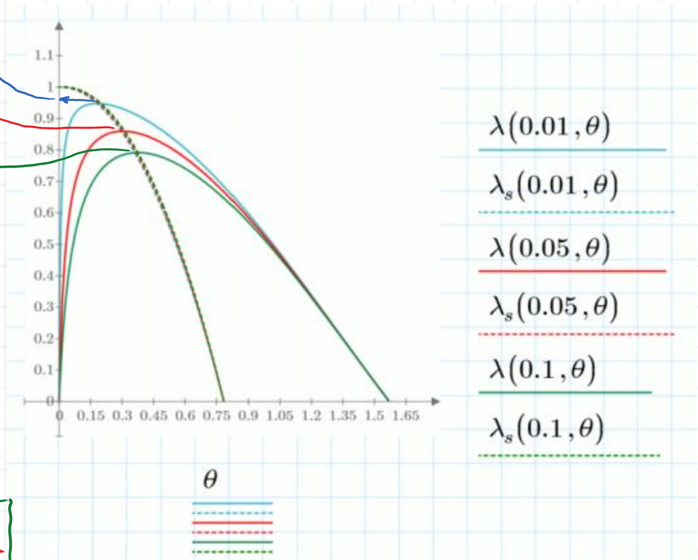
$\text{Find}(\theta_g) = 0.181$

$\varepsilon := 0.01$

$\lambda(0.01, 0.181) = 0.948$

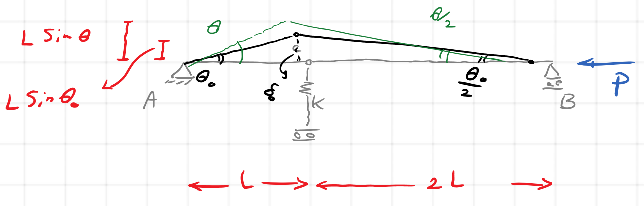
$\lambda(0.05, 0.303) = 0.859$

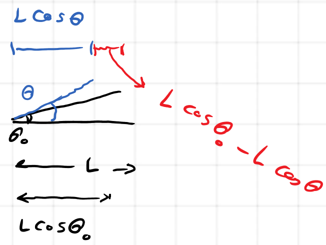
$\lambda(0.1, 0.376) = 0.792$



$\varepsilon = 0.05 \rightarrow \lambda_{cr} = 0.859$

$\hookrightarrow P/P_{cr} = \frac{2}{3}kL$ $\Big\} \rightarrow$ $\boxed{P_{cr} = 0.859 \times \frac{2}{3}kL}$





$$\Delta HB = L\cos\theta_0 - L\cos\theta + 2L\cos\frac{\theta_0}{2} - 2L\cos\frac{\theta}{2}$$

$$\Delta HB = L\left(\cos\theta_0 - \cos\theta + 2\cos\frac{\theta_0}{2} - 2\cos\frac{\theta}{2}\right)$$

$$V = -P\cdot\Delta HB = -PL\left(\cos\theta_0 - \cos\theta + 2\cos\frac{\theta_0}{2} - 2\cos\frac{\theta}{2}\right)$$

$$\delta_{spring} = L\sin\theta - L\sin\theta_0$$

$$W = \frac{1}{2}\cdot k\cdot(L\sin\theta - L\sin\theta_0)^2 = \frac{kL^2}{2}(\sin\theta - \sin\theta_0)^2$$

$$\pi(\theta) = W + V = \frac{kL^2}{2}(\sin\theta - \sin\theta_0)^2 - PL\left(\cos\theta_0 - \cos\theta + 2\cos\frac{\theta_0}{2} - 2\cos\frac{\theta}{2}\right)$$

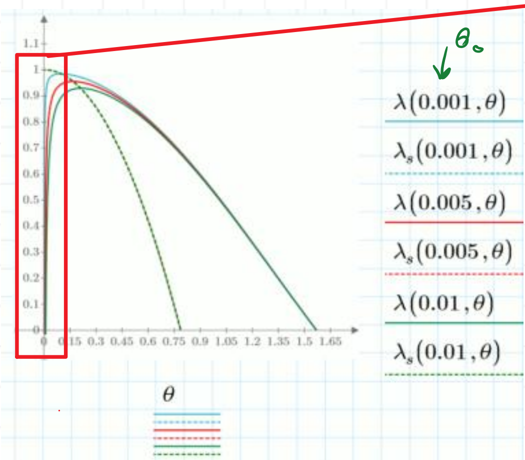
$$\frac{\partial\pi}{\partial\theta} = kL^2(\sin\theta - \sin\theta_0)(\cos\theta) - PL\left(\sin\theta + \sin\frac{\theta}{2}\right) = 0$$

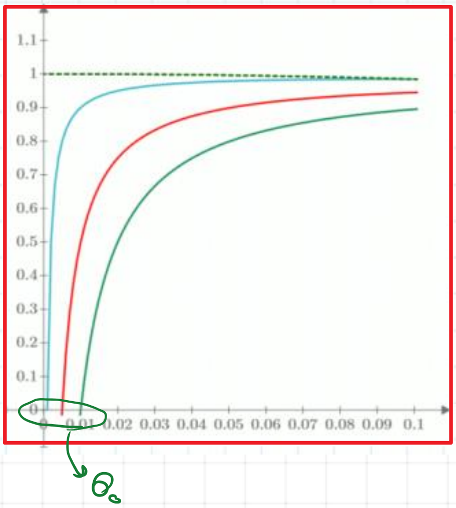
$$\frac{P}{\frac{2}{3}kL} = \frac{kL}{\frac{2}{3}kL}\cdot\frac{\sin\theta\cos\theta - \sin\theta_0\cos\theta}{\sin\theta + \sin\frac{\theta}{2}} \qquad P_0 = \frac{2}{3}kL$$

$$\lambda = \frac{3}{2}\cdot\frac{\sin\theta\cos\theta - \sin\theta_0\cos\theta}{\sin\theta + \sin\theta/2}$$

$$\frac{\partial^2\pi}{\partial\theta^2} = kL^2\Big[\underbrace{\cos\theta\cdot\cos\theta - (\sin\theta - \sin\theta_0)\sin\theta}_{\underbrace{\cos^2\theta - \sin^2\theta}_{\cos 2\theta} + \sin\theta\sin\theta_0}\Big] - PL\left(\cos\theta + \frac{1}{2}\cos\frac{\theta}{2}\right) > 0$$

$$\lambda\ \frac{PL}{\frac{2}{3}kL^2} < \frac{kL^2}{\frac{2}{3}kL^2}\cdot\frac{\cos 2\theta + \sin\theta\sin\theta_0}{\cos\theta + \frac{1}{2}\cos\frac{\theta}{2}} \rightarrow \lambda < \frac{3}{2}\cdot\frac{\cos 2\theta + \sin\theta\sin\theta_0}{\cos\theta + \frac{1}{2}\cos\theta/2}$$





$\theta_0 := 0.01$

$\lambda(0.001, 0.098) = 0.985$

$\lambda(0.005, 0.167) = 0.955$

$\lambda(0.01, 0.212) = 0.929$

Guess Values

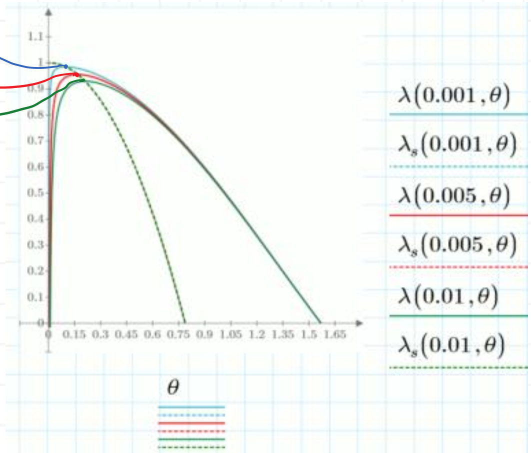
$\theta_g := 0.3$

Constraints

$\lambda(\theta_0, \theta_g) = \lambda_s(\theta_0, \theta_g)$

Solver

Find $(\theta_g) = 0.212$



$\theta_0 = 0.01^{rad} \rightarrow \left.\begin{array}{l} \lambda_{cr} = 0.929 \\ P_0 = \frac{2}{3} kL \end{array}\right\} \Rightarrow P_{cr} = \frac{2}{3} \times 0.929\, kL = \boxed{0.62\, kL}$