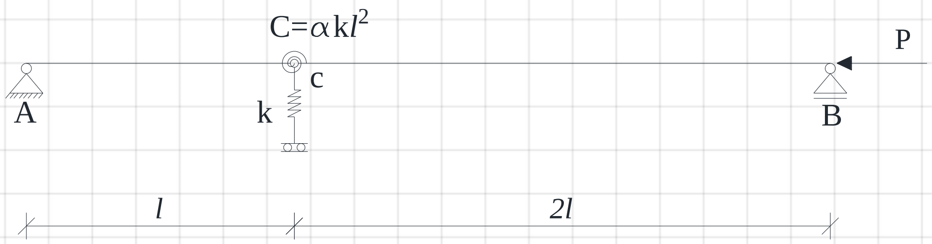
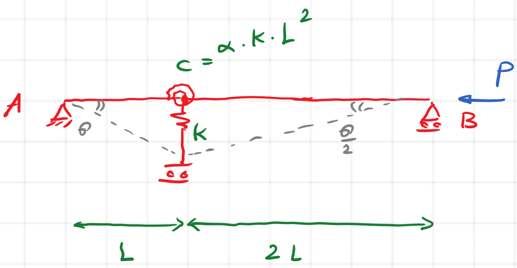
In the other examples, the equilibrium path of the shown system has been determined. The first one was with a rotational spring, and the system was stable after the rotation of the bars. However, in the translational spring, the system was unstable. In this example, both springs are combined, and the target is to calculate the minimum rotational stiffness of the spring to stabilize the system. Assume the rotational stiffness C can be a portion of kl^2, e.g., $C = \alpha k l^2$.





$$M = K\cdot\theta$$

rotational spring coefficient

$$K_\theta = \frac{kN\cdot m}{rad}$$

$$F = k\cdot\delta \Rightarrow k = \frac{kN}{m}$$

$$\frac{K_\theta}{k} = \frac{kN\cdot m/rad}{kN/m} = m^2/rad$$

$$K_\theta = c \Rightarrow \frac{c}{k} = \frac{m^2}{rad} = \alpha\cdot L^2$$

$\alpha \rightarrow$ constant

$$\boxed{c = \alpha\cdot k\cdot L^2}$$

$$\Delta_{HB} = L - L\cos\theta + 2L - 2L\cos\left(\frac{\theta}{2}\right)$$

$$V = -P\cdot\Delta_{HB} = -PL\left(1-\cos\theta+2-2\cos\left(\frac{\theta}{2}\right)\right)$$

$$W = \frac{1}{2}k\cdot\delta^2 + \frac{1}{2}c\,\theta_r^2$$

$$\delta = L\sin\theta$$

$$\theta_r = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$$

$$W = \frac{1}{2}k(L\sin\theta)^2 + \frac{1}{2}c\left(\frac{3}{2}\theta\right)^2 = \frac{kL^2}{2}\sin^2\theta + \frac{9}{8}c\theta^2$$

$$\pi(\theta) = W + V = \frac{kL^2}{2}\sin^2\theta + \frac{9}{8}c\theta^2 - PL\left(1-\cos\theta + 2 - 2\cos\frac{\theta}{2}\right)$$

1) $\sin\theta \simeq \theta$, $\theta\to 0$

$1-\cos\theta = \frac{1}{2}\theta^2$, $\theta\to 0$

$$\rightarrow \pi(\theta) = \frac{kL^2\theta^2}{2} + \frac{9}{8}c\theta^2 - PL\left(\frac{1}{2}\theta^2 + \left(\frac{\theta}{2}\right)^2\right)$$

$$\pi(\theta) = \frac{kL^2}{2}\theta^2 + \frac{9}{8}c\theta^2 - \frac{3}{4}PL\theta^2$$

$$\frac{\partial\pi}{\partial\theta} = kL^2\theta + \frac{9}{4}\underset{\downarrow\ \alpha kL^2}{c}\theta - \frac{3}{2}PL\theta = 0 \Rightarrow \begin{cases}\theta = 0\\ P = \frac{2}{3}kL\left(1+\frac{9}{4}\alpha\right)\end{cases}$$

$$\alpha = 0 \Rightarrow \boxed{P_0 = \frac{2}{3}kL}$$

2) $$\pi(\theta) = \frac{kL^2}{2}\sin^2\theta + \frac{9}{8}\alpha kL^2\theta^2 - PL\left(1-\cos\theta+2-2\cos\frac{\theta}{2}\right)$$

$$\frac{\partial\pi}{\partial\theta} = kL^2\sin\theta\cos\theta + \frac{9}{4}\alpha kL^2\theta - PL\left(\sin\theta + \sin\frac{\theta}{2}\right) = 0$$

$$\lambda\ \frac{P_{cr}}{\frac{2}{3}kL} = \frac{kL}{\frac{2}{3}kL}\left(\frac{\sin\theta\cos\theta + \frac{9}{4}\alpha\theta}{\sin\theta + \sin\frac{\theta}{2}}\right) \Rightarrow \lambda = \frac{3}{2}\cdot\frac{\sin\theta\cos\theta + \frac{9}{4}\alpha\theta}{\sin\theta + \sin\frac{\theta}{2}}$$

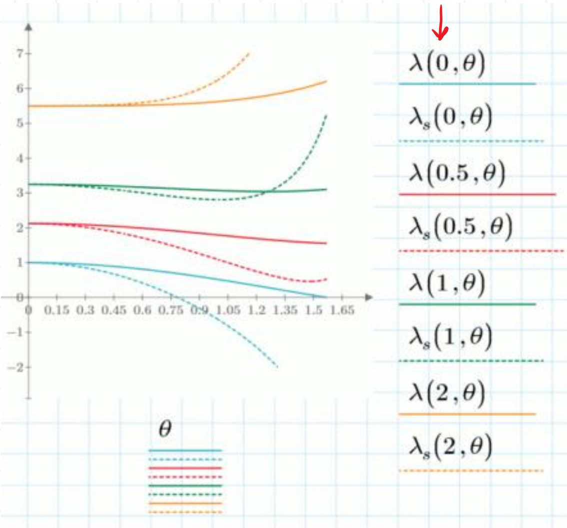
$$\frac{\partial \Pi}{\partial \theta} = kL^2 \sin\theta\cos\theta + \frac{9}{4}\alpha kL^2\theta - PL\left(\sin\theta + \sin\frac{\theta}{2}\right)$$

$$\frac{\partial^2 \Pi}{\partial \theta^2} = kL^2\left(\underbrace{\cos\theta\cos\theta - \sin\theta\sin\theta}_{\cos 2\theta}\right) + \frac{9}{4}\alpha\, kL^2 - PL\left(\cos\theta + \frac{1}{2}\cos\frac{\theta}{2}\right) > 0$$

$$\lambda\ \frac{P}{\frac{2}{3}kL} < \frac{kL}{\frac{2}{3}kL}\left(\frac{\cos 2\theta + \frac{9}{4}\alpha}{\cos\theta + \frac{1}{2}\cos\frac{\theta}{2}}\right) \Rightarrow \lambda < \frac{3}{2}\cdot\frac{\cos 2\theta + \frac{9}{4}\alpha}{\cos\theta + \frac{1}{2}\cos\frac{\theta}{2}}$$

$$\lambda = \frac{3}{2}\cdot\frac{\sin\theta\cos\theta + \frac{9}{4}\alpha\theta}{\sin\theta + \sin\frac{\theta}{2}}$$

α



when α is zero the system is unstable

by increasing α the system is getting to be stable.

what is the minimum value of α that the system behavior changes from unstable to stabe.

$$\lambda = \frac{3}{2}\cdot\frac{\sin\theta\cdot\cos\theta + \frac{9}{4}\alpha\cdot\theta}{\sin\theta + \sin\frac{\theta}{2}} \qquad \lambda < \frac{3}{2}\cdot\frac{\cos(2\theta) + \frac{9}{4}\alpha}{\cos\theta + \frac{1}{2}\cos\left(\frac{\theta}{2}\right)}$$

$$\cancel{\frac{3}{2}}\ \frac{\overbrace{\sin\theta\cdot\cos\theta + \frac{9}{4}\alpha\cdot\theta}^{A}}{\underbrace{\sin\theta + \sin\frac{\theta}{2}}_{B}} < \cancel{\frac{3}{2}}\cdot\frac{\overbrace{\cos(2\theta) + \frac{9}{4}\alpha}^{C}}{\underbrace{\cos\theta + \frac{1}{2}\cos\frac{\theta}{2}}_{D}}$$

$$\frac{A}{B} < \frac{C}{D} \Rightarrow \frac{C}{D} - \frac{A}{B} > 0 \Rightarrow F(\alpha,\theta) = \frac{C}{D} - \frac{A}{B} \Rightarrow \text{minimum } F\Big|_{\alpha,\theta} = 0$$

$$\hookrightarrow \frac{\partial F}{\partial \theta} = 0$$

$$\begin{cases} F = 0 \\ \dfrac{\partial F}{\partial \theta} = 0 \end{cases} \qquad F = \frac{C}{D} - \frac{A}{B} = \frac{B \cdot C - A \cdot D}{B \cdot D} = 0 \Rightarrow \boxed{B \cdot C - A \cdot D = 0}$$

$$\frac{\partial F}{\partial \theta} = \frac{\left[\left(\frac{\partial B}{\partial \theta} \cdot C + \frac{\partial C}{\partial \theta} \cdot B\right) - \left(\frac{\partial A}{\partial \theta} \cdot D + \frac{\partial D}{\partial \theta} \cdot A\right)\right] \cdot B \cdot D - \cancel{\left(\frac{\partial B}{\partial \theta} \cdot D + \frac{\partial D}{\partial \theta} \cdot B\right) \cdot \left[B \cdot C - A \cdot D\right]}}{(B \cdot D)^2} = 0$$

$$\underbrace{\left[\left(\frac{\partial B}{\partial \theta} \cdot C + \frac{\partial C}{\partial \theta} \cdot B\right) - \left(\frac{\partial A}{\partial \theta} \cdot D + \frac{\partial D}{\partial \theta} \cdot A\right)\right]}_{\boxed{\frac{\partial (B.C - A.D)}{\partial \theta} \cdot B \cdot D = 0}} \cdot B . D = 0$$

$$\lambda = \frac{A}{B}, \qquad \lambda < \frac{C}{D}$$

Guess Values

$\theta_g := 0.2$ $\alpha_g := 1.5$

Constraints

$F(\theta_g, \alpha_g) = 0$

$G(\theta_g, \alpha_g) = 0$

Solver

$\text{Find}(\theta_g, \alpha_g) = \begin{bmatrix} 0.009 \\ 1.926 \end{bmatrix}$

$\lambda(1.926, 0.001) = 5.334$

$\lambda(0, 0.001) = 1$

$\alpha_{cr} = 1.926$

$\downarrow$

The minimum α that the system behavior will change from unstable to be stable.