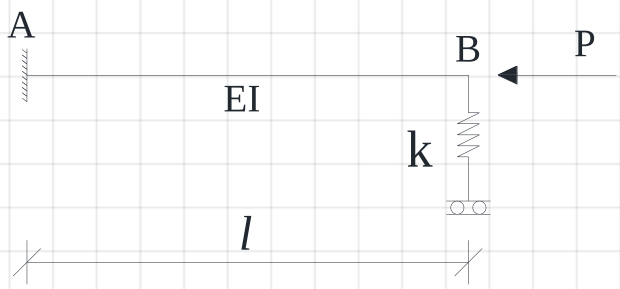
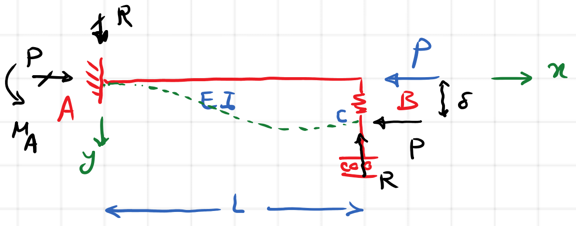
In practical engineering, there are debates in several cases where the buckling shape and mode would not be clear. In such a case, the minimum stiffness of one element can be determined, and the buckling mode can be understood based on that.

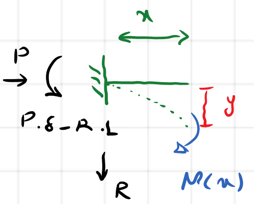
In the figure, a cantilever beam with the bending rigidity of EI is subjected to a compressive load, p. The beam's free end is supported by a translational spring with the constant of c.

Determine the critical value of spring coefficient c so that the buckling shape of the compressive element would change from the free end to a supported one.





$$M_A = P\cdot\delta - R\cdot L$$



$$M(x) = -P\cdot y + R\cdot x + P\cdot\delta - R L$$

$$M = E\cdot I\cdot y''$$

$$\left(EI\, y'' + P y = R\cdot x + P\cdot\delta - R\cdot L\right) \div EI$$

$$y'' + \frac{P}{EI}\cdot y = \frac{R}{EI}x + \frac{P}{EI}\cdot\delta - \frac{R}{EI}\cdot L \qquad \boxed{\frac{P}{EI} = \lambda^2}$$

$$y'' + \lambda^2 y = \frac{R}{EI}x + \lambda^2\cdot\delta - \frac{R}{EI}\cdot L$$

$$y = A\cdot\sin\lambda x + B\cdot\cos\lambda x + \frac{R}{\lambda^2 EI}\cdot x + \delta - \frac{R}{\lambda^2 EI}\cdot L$$

$$y' = A\cdot\lambda\cos\lambda x - B\lambda\sin\lambda x + \frac{R}{\lambda^2 EI}$$

$y(0)=0 \rightarrow 0 = B + \delta - \frac{R}{\lambda^2 EI}L = 0$

$y'(0)=0 \rightarrow 0 = A\cdot\lambda + \frac{R}{\lambda^2 EI}$

$y(L)=\delta \rightarrow A\sin\lambda L + B\cos\lambda L + \frac{R}{\lambda^2 EI}L + \delta - \frac{R}{\lambda^2 EI}L = \delta$

$$R = c\cdot\delta \rightarrow \boxed{\delta = \frac{R}{c}}$$

$$0 = B + \delta - \frac{R}{\lambda^2 EI} L$$

$$0 = A \cdot \lambda + \frac{R}{\lambda^2 EI}$$

$$A \sin\lambda L + B \cos\lambda L + \frac{R}{\lambda^2 EI} L + \cancel{\delta} - \frac{R}{\lambda^2 EI} L = \cancel{\delta}$$

$$\boxed{\delta = \frac{R}{c}}$$

$$\begin{cases} B + \dfrac{R}{c} - \dfrac{R}{\lambda^2 EI} \cdot L = 0 \\ A\lambda + \dfrac{R}{\lambda^2 EI} = 0 \end{cases}$$

$$A \sin\lambda L + B \cos\lambda L + \cancel{\frac{R}{\lambda^2 EI}} L - \cancel{\frac{R}{\lambda^2 EI}} \cdot L = 0$$

$$\begin{cases} 0A + 1B + \left(\dfrac{1}{c} - \dfrac{L}{\lambda^2 EI}\right) R = 0 \\ \lambda A + 0B + \dfrac{1}{\lambda^2 EI} \cdot R = 0 \\ \sin\lambda L \cdot A + \cos\lambda L \cdot B + 0R = 0 \end{cases}$$

$$\begin{vmatrix} 0 & 1 & \dfrac{1}{c} - \dfrac{L}{\lambda^2 EI} \\ \lambda & 0 & \dfrac{1}{\lambda^2 EI} \\ \sin\lambda L & \cos\lambda L & 0 \end{vmatrix} = 0 \Rightarrow$$

$$-1\left(\cancel{\lambda \times 0} - \frac{\sin\lambda L}{\lambda^2 EI}\right) + \left(\frac{1}{c} - \frac{L}{\lambda^2 EI}\right)\left(\lambda \cdot \cos\lambda L - \cancel{0 \sin\lambda L}\right) = 0$$

$$\frac{\sin\lambda L}{\lambda^2 EI} + \frac{\lambda \cos\lambda L}{c} - \frac{\lambda L \cos\lambda L}{\lambda^2 EI} = 0 \Rightarrow \frac{\sin\lambda L - \lambda L \cos\lambda L}{EI \cdot \lambda^2} = -\frac{\lambda \cos\lambda L}{c}$$

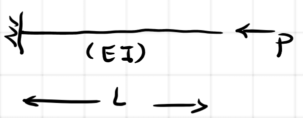
$$\frac{\sin\lambda L - \lambda L\cos\lambda L}{EI\cdot\lambda^2} = -\frac{\lambda\cos\lambda L}{c} \Rightarrow \left(\lambda L\cdot\cos\lambda L - \sin\lambda L = \frac{EI}{c}\cdot\lambda^3\cdot\cos\lambda L\right) \div \cos\lambda L$$

$$\lambda L - \tan\lambda L = \frac{EI}{c}\cdot\frac{\lambda^3\cdot L^3}{L^3} \qquad \lambda^2 = \frac{P}{EI} \quad \frac{kN}{\frac{kN}{m^2}\cdot m^4} \qquad \lambda^2 \equiv \frac{1}{m^2}$$

$$\lambda L - \tan\lambda L = \frac{EI}{cL^3}\cdot(\lambda L)^3 \Longrightarrow$$

$$\frac{cL^3}{EI} = \frac{(\lambda L)^3}{\lambda L - \tan(\lambda L)} \qquad \text{(I)}$$

if $c=0$



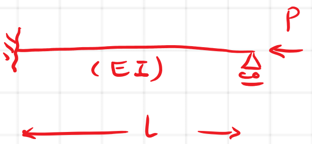
$$P_E = \frac{\pi^2 EI}{4L^2}$$

$$\text{(I)} \rightarrow c=0 \longrightarrow \begin{cases} \lambda L = 0 \rightarrow \lambda = 0 \rightarrow P = 0 \\ \lambda L - \tan(\lambda L) = \pm\infty \end{cases}$$

$$\tan(\lambda L) = \infty \Rightarrow \lambda L = \frac{\pi}{2}$$

$$\rightarrow \lambda = \frac{\pi}{2L} \Rightarrow \lambda^2 = \frac{\pi^2}{4L^2} \Rightarrow \frac{P}{EI} = \frac{\pi^2}{4L^2} \Rightarrow \boxed{P_{cr} = \frac{\pi^2 EI}{4L^2}}$$

if $c=\infty$



$$\text{(I)} \rightarrow c=\infty \Rightarrow \lambda L - \tan\lambda L = 0 \Rightarrow$$

$L_{eff} = 0.7L$

$$\rightarrow P_E = \frac{\pi^2 EI}{(0.7L)^2} = \frac{\pi^2 EI}{0.49L^2}$$

Solver Constraints / Guess Values

$\lambda l_g := 2$

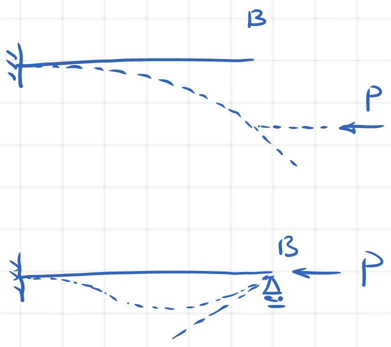
$\lambda l_g - \tan(\lambda l_g) = 0$

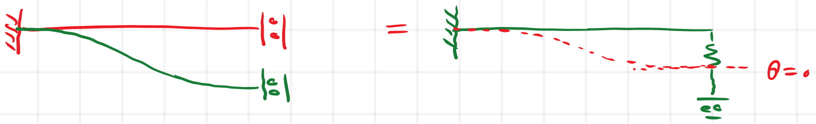
$\text{Find}(\lambda l_g) = 4.493$

$$\lambda L = 4.493 \Rightarrow \lambda = \frac{4.493}{L} \Rightarrow \lambda^2 = \frac{4.493^2}{L^2} \Rightarrow \frac{P_E}{EI} = \frac{20.187\,EI}{L^2}\cdot\frac{\pi^2}{\pi^2}$$

$$P_E = \frac{\pi^2 EI}{\frac{\pi^2}{20.187}\cdot L^2} = \frac{\pi^2 EI}{0.49L^2} = \frac{\pi^2 EI}{(0.7L)^2} \quad \text{(ok)}$$

$$\frac{cL^3}{EI} = \frac{(\lambda L)^3}{\lambda L - \tan(\lambda L)}$$

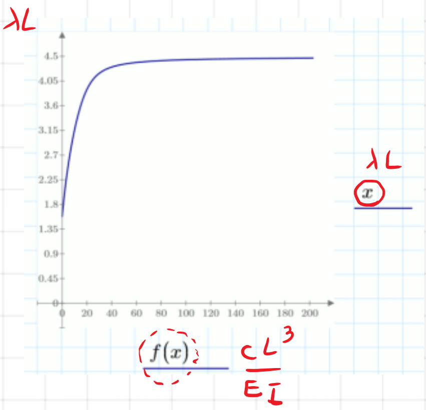




$L_{eff} = L$

$$P_{cr} = \frac{\pi^2 EI}{L^2} \Rightarrow \frac{P_{cr}}{EI} = \lambda^2 = \frac{\pi^2}{L^2} \Rightarrow \boxed{\lambda L = \pi}$$

$$\frac{cL^3}{EI} = \frac{\pi^3}{\pi - \underbrace{\tan \pi}_{0}} = \pi^2 \Rightarrow \boxed{C_{cr} = \frac{\pi^2 EI}{L^3}}$$



Design properties of hot finished Circular Hollow Section (CHS) for S355 steel class (γ_{M0} =

		Profile dimensions		Area properties				
Profile	Drawing	Ext. Diameter D [mm]	Wall thickness t [mm]	Weight m [kg/m]	External perimeter P [m]	Area A [mm^2]	Shear area A_v [mm^2]	Second moment of area I [$\times 10^6$ mm^4]
CHS 168.3 / 10	dxf	168.3	10.0	39.0	0.529	4973	3166	15.64

CHS 168.3/10

$$C = \frac{\pi^2 EI}{L^3}, \quad L = 5m, \quad E = 210\,GPa, \quad I = 15.64 \times 10^6\ mm^4$$

$$C_{cr} = \frac{\pi^2 \times 210\,GP \times 15.6 \times 10^6\ mm^4}{(5m)^3} = 258.7\ KN/m$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 210\,GP \times 15.6 \times 10^6\ mm^4}{(5m)^2} = 1293\ kN$$

$$P = 500\ kN \rightarrow \text{load factor}: \frac{1293\ kN}{500\ kN} = 2.586$$