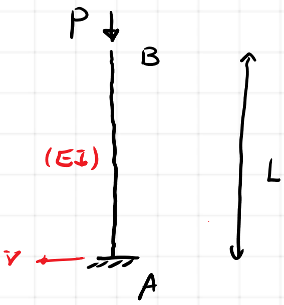
In another video, we learned about the stability of one rigid element and its behavior after bifurcation.

The rotational spring can be achieved from the bending stiffness of the component in terms of strain energy.

Suppose a cantilever is subjected to a compressive force at its tip.

a) Determine the total potential energy of the compressive element.
b) Find out the bifurcation load.
c) If the load increases after approaching its bifurcation load, determine the post-buckling load path and its stability state.
d) Determine the maximum horizontal displacement of the tip.
e) Determine the shortening length of the beam based on the post-buckling analysis.
f) Check your calculation with a numerical solution and FEM software.




$$\Pi = \frac{1}{2}\int_0^L E\cdot I\cdot \kappa^2\, dx - \frac{1}{2}P\int_0^L v'^2\, dx$$

$$\kappa = -\frac{v''}{\sqrt{1-v'^2}} \times \frac{\sqrt{1+v'^2}}{\sqrt{1+v'^2}} = -\frac{v''\cdot\sqrt{1+v'^2}}{\sqrt{1-\underbrace{v'^4}_{0}}}$$

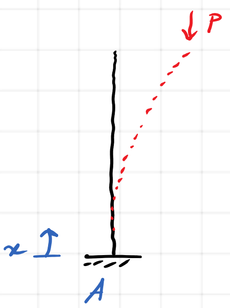
$v$ = displacement

$v'$ = rotation

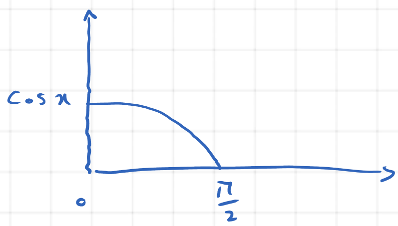
taylor: $\sqrt{1+v'^2} = 1 + \frac{1}{2}v'^2$

$$\kappa = -v''\cdot\left(1+\frac{1}{2}v'^2\right)$$

$$\Pi = \frac{1}{2}\int_0^L E\cdot I\cdot v''^2\cdot\left(1+\frac{1}{2}v'^2\right)^2 dx - \frac{P}{2}\int v'^2\, dx$$




$$P_{cr} = \frac{\pi^2 EI}{(2L)^2}$$




$$v(x) = A\left(\cos\left(\frac{\pi}{2L}x\right) - 1\right)$$

$$\boxed{A = \alpha\cdot L}$$

$$v(x) = \alpha\cdot L\cdot\left(\cos\left(\frac{\pi}{2L}\cdot x\right) - 1\right)$$

$$\Pi(E,I,p,l,\alpha) := \frac{1}{2}\cdot\int_0^l E\cdot I\cdot\left(\frac{d^2}{dx^2}v(x,l,\alpha)\right)^2\cdot\left(1+\frac{1}{2}\cdot\left(\frac{d}{dx}v(x,l,\alpha)\right)^2\right)^2 dx$$
$$-\frac{p}{2}\cdot\int_0^l\left(\frac{d}{dx}v(x,l,\alpha)\right)^2 dx$$

$$\Pi(E,I,p,l,\alpha) \xrightarrow{expand} \frac{\alpha^6\cdot E\cdot I\cdot\pi^8}{32768\cdot l}+\frac{\alpha^4\cdot E\cdot I\cdot\pi^6}{1024\cdot l}+\frac{\alpha^2\cdot E\cdot I\cdot\pi^4}{64\cdot l}-\frac{\alpha^2\cdot l\cdot p\cdot\pi^2}{16}$$

$$\Pi = \frac{\pi^2 EI}{L}\left[\frac{(\alpha\pi)^6}{32768}+\frac{(\alpha\pi)^4}{1024}+\frac{(\alpha\pi)^2}{64}\right]-\alpha^2\frac{\pi^2}{16}\cdot P\cdot L$$

$$\frac{d}{d\alpha}\Pi(E,I,p,l,\alpha) \xrightarrow{expand} \frac{3\cdot\alpha^5\cdot E\cdot I\cdot\pi^8}{16384\cdot l}+\frac{\alpha^3\cdot E\cdot I\cdot\pi^6}{256\cdot l}+\frac{\alpha\cdot E\cdot I\cdot\pi^4}{32\cdot l}-\frac{\alpha\cdot l\cdot p\cdot\pi^2}{8}$$

$$\frac{d^2}{d\alpha^2}\Pi(E,I,p,l,\alpha) \xrightarrow{expand} \frac{15\cdot\alpha^4\cdot E\cdot I\cdot\pi^8}{16384\cdot l}+\frac{3\cdot\alpha^2\cdot E\cdot I\cdot\pi^6}{256\cdot l}+\frac{E\cdot I\cdot\pi^4}{32\cdot l}-\frac{l\cdot p\cdot\pi^2}{8}$$

$$\frac{\partial\Pi}{\partial\alpha} = \frac{EI\pi^3}{L}\left[\frac{3}{16384}(\pi\alpha)^5+\frac{1}{256}(\pi\alpha)^3+\frac{1}{32}(\pi\alpha)\right]-\frac{\alpha\pi^2}{8}P\cdot L = 0$$

$$P_{cr} = \frac{\pi EI}{\alpha L^2}\cdot 8\left[\frac{3}{16384}(\pi\alpha)^5+\frac{1}{256}(\pi\alpha)^3+\frac{1}{32}(\pi\alpha)\right]$$

$$P_{cr} = \frac{4}{4}\cdot\underbrace{\frac{\pi^2\cancel{\alpha}\cdot EI}{\cancel{\alpha}L^2}}_{4P_{cr_0}}\cdot\left[\frac{3}{2048}(\pi\alpha)^4+\frac{1}{32}(\pi\alpha)^2+\frac{1}{4}\right]$$

$$P_{cr_0} = \frac{\pi^2 EI}{4L^2}$$

$$\frac{P_{cr}}{P_{cr_0}} = \lambda = \frac{3}{512}(\pi\alpha)^4+\frac{1}{8}(\pi\alpha)+1$$

$$\boxed{\lambda = \frac{3}{512}(\pi\alpha)^4 + \frac{1}{8}(\pi\alpha)^2 + 1}$$

$$\frac{d^2}{d\alpha^2}\Pi(E,I,p,l,\alpha) \xrightarrow{expand} \frac{15\cdot\alpha^4\cdot E\cdot I\cdot\pi^8}{16384\cdot l} + \frac{3\cdot\alpha^2\cdot E\cdot I\cdot\pi^6}{256\cdot l} + \frac{E\cdot I\cdot\pi^4}{32\cdot l} - \frac{l\cdot p\cdot\pi^2}{8}$$

$$\frac{\partial^2\pi}{\partial\alpha^2} = \frac{\pi^4 EI}{L}\left[\frac{15}{16384}(\pi\alpha)^4 + \frac{3}{256}(\pi\alpha)^2 + \frac{1}{32}\right] - \frac{P\cdot\pi^2\cdot L}{8} > 0$$

$$P_{cr} < 8\underbrace{\frac{\pi^2 EI}{L^2}}_{4P_0}\overbrace{\left[\frac{15}{16384}(\pi\alpha)^4 + \frac{3}{256}(\pi\alpha)^2 + \frac{1}{32}\right]}^{\lambda_s}$$

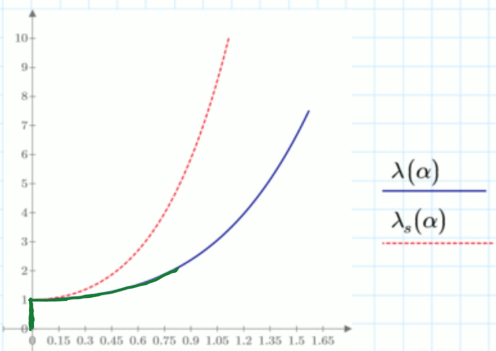
$$\frac{P_{cr}}{P_0} = \lambda < \boxed{\frac{15}{512}(\pi\alpha)^4 + \frac{3}{8}(\pi\alpha)^2 + 1}$$

The equation valids for any $\pi\alpha$ value.

$$v = \alpha\cdot L\left[\cos\left(\frac{\pi x}{2L}\right) - 1\right]$$

$$v(x=L) = \alpha\cdot L\left[\cos\frac{\pi}{2} - 1\right]$$

$$\boxed{v(x=L) = -\alpha\cdot L}$$




$$\alpha = \frac{\delta_{tip}}{L}$$

$$\lambda = \frac{3}{512}(\pi\alpha)^4 + \frac{1}{8}(\pi\alpha)^2 + 1$$

$$\frac{3}{512}(\pi\alpha)^4 + \frac{1}{8}(\pi\alpha)^2 + 1 - \lambda = 0 \qquad (\pi\alpha)^2 = X > 0$$

$$\frac{3}{512}x^2 + \frac{1}{8}x + 1 - \lambda = 0$$

$$x = \frac{-\frac{1}{8} + \cancel{\pm}\sqrt{\frac{1}{64} - 4\cdot\frac{3}{512}(1-\lambda)}}{2\times\frac{3}{512}} = \frac{-\frac{1}{8} + \frac{1}{8}\sqrt{1 - \frac{3}{2}(1-\lambda)}}{\frac{3}{256}}$$

$$(\pi\alpha)^2 = \frac{\frac{1}{8}}{\frac{3}{256}}\left[\sqrt{1 - \frac{3}{2}(1-\lambda)} - 1\right] = \frac{32}{3}\left[\sqrt{1 - \frac{3}{2}(1-\lambda)} - 1\right]$$

$$\lambda = \frac{P_{cr}}{P_0 \rightarrow P_E} = \frac{P_E + \Delta P}{P_E} = 1 + \frac{\Delta P}{P_E} \rightarrow 1 - \lambda = -\frac{\Delta P}{P_E}$$

$$(\pi\alpha)^2 = \frac{32}{3}\left[\sqrt{1 + \frac{3}{2}\frac{\Delta P}{P_E}} - 1\right] = \frac{32}{3}\left[1 + \frac{3}{4}\frac{\Delta P}{P_E} - 1\right] = 8\frac{\Delta P}{P_E}$$

$\Delta P \rightarrow 0$ $\rightarrow$ $\sqrt{1+x} = 1 + \frac{1}{2}x$

$x \rightarrow 0$

$$\pi\alpha = \sqrt{8\frac{\Delta P}{P_E}} = 2\sqrt{\frac{2\Delta P}{P_E}} \rightarrow \alpha = \frac{2}{\pi}\sqrt{\frac{2\Delta P}{P_E}}$$

$$\delta_{max} = \alpha \cdot L = \frac{2L}{\pi}\sqrt{\frac{2\Delta P}{P_E}}$$

$$\delta_{max} = \frac{2L}{\pi}\sqrt{\frac{2\Delta P}{P_E}}$$

$\Delta_{max}$


$$M_{max} = P \cdot \Delta_{max} = (P_E + \Delta P) \cdot \frac{2L}{\pi}\sqrt{\frac{2\Delta P}{P_E}}$$

$M_{max}$

$$\sigma = \frac{P}{A} + \frac{M}{W}$$


$$U = \frac{1}{2}\int_0^L v'^2 \, dx$$

$$v = A\left[\cos\left(\frac{\pi x}{2L}\right) - 1\right] = \delta_{max} \cdot \left[\cos\left(\frac{\pi x}{2L}\right) - 1\right]$$

$$v' = \delta_{max}\left[-\frac{\pi}{2L} \cdot \sin\frac{\pi x}{2L}\right] \Rightarrow v' = \frac{\cancel{2L}}{\cancel{\pi}}\sqrt{\frac{2\Delta P}{P_E}} \cdot -\frac{\cancel{\pi}}{\cancel{2L}} \cdot \sin\frac{\pi x}{2L}$$

$$v'^2 = \frac{2\Delta P}{P_E} \cdot \sin^2\frac{\pi x}{2L}$$

$$U = \frac{1}{2} \cdot \frac{2\Delta P}{P_E}\int_0^L \sin^2\left(\frac{\pi x}{2L}\right) = \frac{\Delta P}{P_E} \cdot \int_0^L \sin^2\left(\frac{\pi x}{2L}\right)$$

$$\int_0^l \sin\left(\frac{\pi \cdot x}{2\,l}\right)^2 dx \rightarrow \frac{l}{2}$$

$$U = \frac{\Delta P}{P_E} \cdot \frac{L}{2} = \frac{1}{2}\frac{\Delta P}{P_E} \cdot L$$

CHC 168.3x10

$L = 5\,m$

$I = 15.64 \times 10^{6}\ mm^{4}$

$E = 210\ GPa$

| Profile | Drawing | Ext. Diameter D [mm] | Wall thickness t [mm] | Weight m [kg/m] | External perimeter P [m] | Area A [mm²] | Shear area $A_v$ [mm²] | Second moment of area I [×10⁶ mm⁴] |
|---|---|---|---|---|---|---|---|---|
| CHS 168.3 / 10 | dxf | 168.3 | 10.0 | 39.0 | 0.529 | 4973 | 3166 | 15.64 |

$\underbrace{S355}$

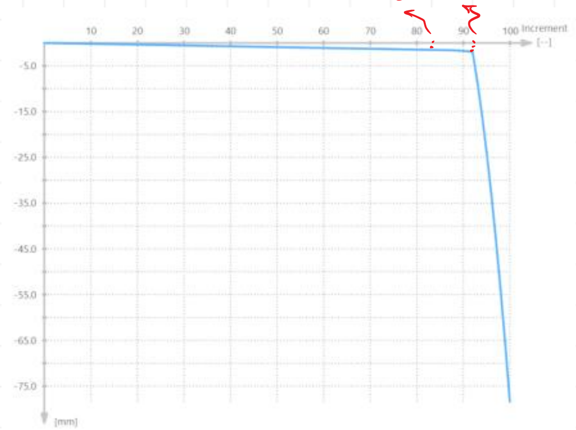
$f_y = 355\ MPa$

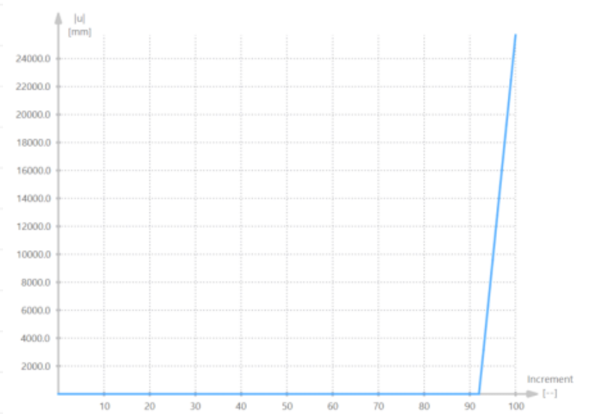
$$P_E = \frac{\pi^2 E I}{4L^2} = 324\ kN$$

$$\sigma_E = \frac{P_E}{A} = \frac{324\,kN}{4973\,mm^2} = 65\ MPa$$

85 93

(u)




$P_E < P < \underbrace{P_E + \Delta P}$

324 93 step

$\underbrace{0.93 \times 380}_{353}$

$\boxed{\Delta P = 29\ kN}$

$\frac{29}{324} = 10\%$