In another video, we considered the effect of imperfection on stability. As noticed, the buckling load is not changed but losing stability, and the pattern of failure due to imperfection is altered. The solved example was for a rigid bar, though.

In this example, we will learn how to apply the imperfection on a compressive element. Such an effect is called the amplification factor that can be seen directly in some codes like EN 1992-1-1 clause 5.8.7.3(4) or can be presented indirectly as imperfection buckling curves like EN 1993-1-1 clause 6.3.1.2.

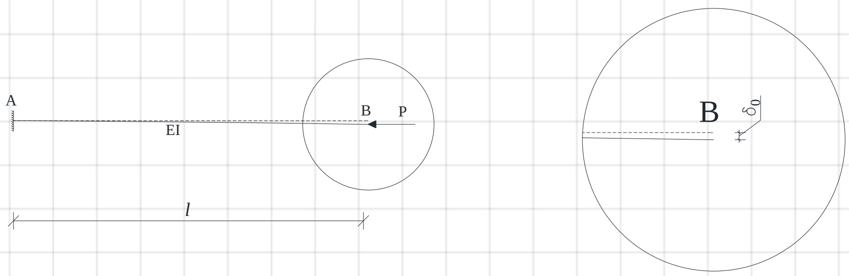
Suppose a cantilever compressive element is initially imperfect with an approximate imperfection function:

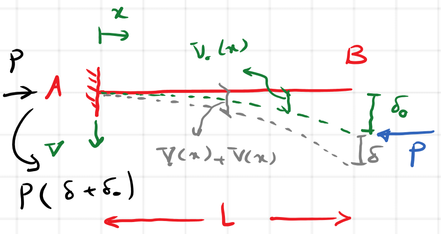
$$v_0(x) = \delta_0 \left(1 - \cos\left(\frac{\pi . x}{2l}\right)\right)$$

Where: δ_0 is the deflection of the beam tip, l is the element's length and $v_0(x)$ represents the initial deflection at the distance x from the support A.

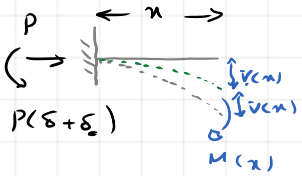
The element is subjected to the compressive force p.

a) Determine the maximum deflection of the beam considering the initial deformation.
b) Determine the maximum bending moment based on the applied load p.
c) Determine the amplification factor.
d) For a circular hollow section, determine the maximum stress in the element if $\delta_0 = \frac{l}{1000}$.
e) Apply the bow imperfection according to EN 1993-1-1, determine the maximum force of p based on the amplification factor derived from the solution, and compare it to EN 1993-1-1.





$$v_0(x) = \delta_0\left(1-\cos\left(\frac{\pi x}{2L}\right)\right)$$



$$M(x) = -P\cdot(v(x)+v_0(x)) + P(\delta+\delta_0)$$

$$EI\,v''(x) = -P\,v(x) - P\,v_0(x) + P\cdot\delta + P\cdot\delta_0$$

$$EI\,v''(x) = -P\,v(x) - P\cdot\delta_0\left(1-\cos\left(\frac{\pi x}{2L}\right)\right) + P\cdot\delta + P\cdot\delta_0$$

$$EI\,v''(x) = -P\cdot v(x) - \cancel{P\cdot\delta_0} + P\delta_0\cos\left(\frac{\pi x}{2L}\right) + P\cdot\delta + \cancel{P\cdot\delta_0}$$

$$\left(EI\,v''(x) + P\cdot v(x) = P\cdot\delta_0\cdot\cos\left(\frac{\pi x}{2L}\right) + P\cdot\delta\right) \div EI$$

$$v''(x) + \frac{P}{EI}v(x) = \frac{P}{EI}\delta_0\cdot\cos\left(\frac{\pi x}{2L}\right) + \frac{P}{EI}\cdot\delta \qquad \boxed{\frac{P}{EI}=\lambda^2}$$

$$v''(x)+\lambda^2 v(x) = \underbrace{\lambda^2\cdot\delta_0\cdot\cos\left(\frac{\pi x}{2L}\right)}_{v_2(x)} + \underbrace{\lambda^2\cdot\delta}_{v_1(x)}$$

$$v''(x)+\lambda^2 v(x)=0$$

$$r^2+\lambda^2=0 \Rightarrow \boxed{r=\pm i\lambda} \rightarrow v_h(x) = A\sin\lambda x + B\cos\lambda x$$

$$v_1(x)=\lambda^2\cdot\delta \Longrightarrow v_{p,1}(x) = \frac{\lambda^2\cdot\delta}{\lambda^2}=\delta$$

$$v''(x)+\lambda^2 v(x) = \lambda^2\cdot\delta_0\cdot\cos\left(\frac{\pi x}{2L}\right)$$

$$v_{p,2}(x) = C_1\cdot\cos\left(\frac{\pi x}{2L}\right) + C_2\cdot\sin\left(\frac{\pi x}{2L}\right) \qquad \ast$$

$$v'_{p,2}(x) = -C_1\frac{\pi}{2L}\sin\left(\frac{\pi x}{2L}\right) + C_2\cdot\frac{\pi}{2L}\cdot\cos\left(\frac{\pi x}{2L}\right)$$

$$v''_{p,2}(x) = -C_1\cdot\frac{\pi^2}{4L^2}\cos\left(\frac{\pi x}{2L}\right) - C_2\cdot\frac{\pi^2}{4L^2}\sin\left(\frac{\pi x}{2L}\right) \qquad \ast$$

$$\rightarrow \quad -C_1 \frac{\pi^2}{4L^2} \cos\left(\frac{\pi x}{2L}\right) - C_2 \cdot \frac{\pi^2}{4L^2} \cdot \sin\left(\frac{\pi x}{2L}\right)$$

$$+\lambda^2 \left[C_1 \cos\left(\frac{\pi x}{2L}\right) + C_2 \cdot \sin\left(\frac{\pi x}{2L}\right) \right] = \lambda^2 \cdot \delta_0 \cdot \cos\left(\frac{\pi x}{2L}\right)$$

$$\cos\left(\frac{\pi x}{2L}\right) \underbrace{\left[-C_1 \frac{\pi^2}{4L^2} + C_1 \lambda^2 \right]} + \sin\left(\frac{\pi x}{2L}\right) \underbrace{\left[-C_2 \cdot \frac{\pi^2}{4L^2} + C_2 \lambda^2 \right]} = \underbrace{\lambda^2 \delta_0} \cdot \cos\left(\frac{\pi x}{2L}\right)$$

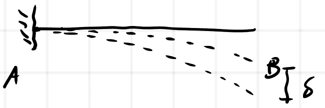
$$\begin{cases} C_1 \lambda^2 - C_1 \dfrac{\pi^2}{4L^2} = \lambda^2 \cdot \delta_0 \longrightarrow C_1 = \dfrac{\lambda^2 \delta_0}{\lambda^2 - \dfrac{\pi^2}{4L^2}} \\ C_2 \lambda^2 - C_2 \dfrac{\pi^2}{4L^2} = 0 \rightarrow \begin{cases} C_2 = 0 \\ \lambda^2 = \dfrac{\pi^2}{4L^2} \rightarrow C_1 = \infty \end{cases} \end{cases}$$

$$\begin{cases} C_2 = 0 \\ C_1 = \dfrac{\lambda^2 \delta_0}{\lambda^2 - \dfrac{\pi^2}{4L^2}} \end{cases}$$

$$V_{P,2}(x) = \frac{\lambda^2 \delta_0}{\lambda^2 - \dfrac{\pi^2}{4L^2}} \cdot \cos\left(\frac{\pi x}{2L}\right)$$

$$V(x) = V_h(x) + V_{P,1}(x) + V_{P,2}(x)$$

$$\underline{\underline{V(x)}} = A \sin \lambda x + B \cos \lambda x + \delta + \frac{\lambda^2 \delta_0}{\lambda^2 - \dfrac{\pi^2}{4L^2}} \cdot \cos\left(\frac{\pi x}{2L}\right)$$



$$V'(x) = A \cdot \lambda \cos \lambda x - B \lambda \sin \lambda x - \frac{\pi}{2L} \cdot \frac{\lambda^2 \cdot \delta_0}{\lambda^2 - \dfrac{\pi^2}{4L^2}} \cdot \sin\left(\frac{\pi x}{2L}\right)$$

$$\begin{cases} V(0) = 0 \rightarrow B + \delta + \dfrac{\lambda^2 \delta_0}{\lambda^2 - \dfrac{\pi^2}{4L^2}} = 0 \quad \text{I} \\ V'(0) = 0 \\ V(L) = \delta \end{cases}$$

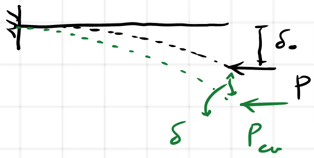
$$A \lambda = 0 \Rightarrow A = 0$$

$$\cancel{A \sin \lambda L} + B \cos \lambda L + \cancel{\delta} + \cancel{\frac{\lambda^2 \cdot \delta_0}{\lambda^2 - \dfrac{\pi^2}{4L^2}} \cdot \cos\left(\frac{\pi}{2}\right)}^{0} = \cancel{\delta}$$

$$B \cdot \cos \lambda L = 0 \Rightarrow \begin{cases} B = 0 \\ \lambda L = \dfrac{\pi}{2} \Rightarrow \times \end{cases} \quad \text{I} \rightarrow \delta = \frac{-\lambda^2 \delta_0}{\lambda^2 - \pi^2 / 4L^2}$$

$$V(x) = \underbrace{A}_{0}\sin\lambda x + \underbrace{B}_{0}\cos\lambda x + \underbrace{\delta}_{\frac{-\lambda^2\delta_0}{\lambda^2 - \frac{\pi^2}{4L^2}}} + \frac{\lambda^2\delta_0}{\lambda^2 - \frac{\pi^2}{4L^2}}\cdot\cos\left(\frac{\pi x}{2L}\right)$$

$$v(x) = \frac{\lambda^2\cdot\delta_0}{\lambda^2 - \frac{\pi^2}{4L^2}}\cdot\left[\cos\left(\frac{\pi x}{2L}\right) - 1\right]$$

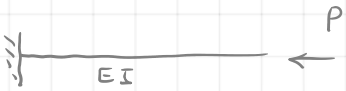


$\delta + \delta_0$

$$\longrightarrow \delta = v(L) = \frac{-\lambda^2\delta_0}{\lambda^2 - \frac{\pi^2}{4L^2}}$$

$$\delta_{max} = \delta + \delta_0 = \frac{-\lambda^2\delta_0}{\lambda^2 - \frac{\pi^2}{4L^2}} + \delta_0 = \frac{\cancel{-\lambda^2\delta_0} + \cancel{\lambda^2\delta_0} - \frac{\pi^2}{4L^2}\cdot\delta_0}{\lambda^2 - \frac{\pi^2}{4L^2}}$$

$$\delta_{max} = \frac{\frac{\pi^2}{4L^2}\cdot\delta_0}{\frac{\pi^2}{4L^2} - \lambda^2}$$



$$\lambda^2 = \frac{P}{EI}, \quad P_{cr} = \frac{\pi^2 EI}{L_{eff}^2} = \frac{\pi^2 EI}{4L^2}$$

$$\delta_{max} = \frac{\frac{\pi^2}{4L^2}\cdot\delta_0}{\frac{\pi^2}{4L^2} - \frac{P}{EI}} = \frac{\delta_0}{1 - \frac{P}{\underbrace{\frac{\pi^2 EI}{4L^2}}_{P_E}}} = \frac{\delta_0}{1 - P/P_E}$$

$$M_{max} = P\cdot\delta_{max} = \frac{P\cdot\delta_0}{1 - \frac{P}{P_E}}$$

$$\sigma = -\frac{P}{A} \pm \frac{M}{W} = -\frac{P}{A} \pm \frac{P \cdot \delta_0}{1-\frac{P}{P_E}} \cdot \frac{1}{W}$$

$$\sigma = -P\left[\frac{1}{A} \mp \frac{1}{1-\frac{P}{P_E}} \cdot \frac{\delta_0}{W}\right]$$

CHS 88.9 / 6.3 , $\boxed{S275}$

$L = 5\,m$, $\delta_0 = \frac{L}{1000}$,

Profile	Drawing	Ext. Diameter D [mm]	Wall thickness t [mm]	Weight m [kg/m]	External perimeter P [m]	Area A [mm²]	Shear area A_v [mm²]	Second moment of area I [×10⁶ mm⁴]	Radius of gyration i [mm]	Elastic section modulus W_{el} [×10³ mm³]	Plastic section modulus W_{pl} [×10³ mm³]
CHS 88.9 / 6.3	dxf	88.9	6.3	12.8	0.279	1635	1041	1.402	29.3	31.55	43.07

$$P_E = P_{cr} = \frac{\pi^2 \cdot EI}{4L^2} = \frac{\pi^2 \times 210\,GPa \times 1.402 \times 10^6\,mm^4}{4 \times (5000\,mm)^2} = 29\,kN$$

$P = 10\,kN$

$$\sigma = -10000(N)\left[\frac{1}{1635(mm^2)} \mp \frac{1}{1-\frac{10\,kN}{29\,kN}} \cdot \frac{\frac{5000\,mm}{1000}}{31.55 \times 10^3 (mm)^3}\right]$$

$$\sigma = -6.12 \pm 2.42 \Rightarrow \begin{cases} -3.72 & MPa \\ -8.54 & MPa \end{cases}$$

Determine the maximum P that material yields.

$$\sigma = -P\left[\frac{1}{A} \mp \frac{1}{1-\frac{P}{P_E}} \cdot \frac{\delta_0}{W}\right]$$

$$\underset{-\sigma}{\sigma} = -\frac{P}{A} - \frac{P}{1-\frac{P}{P_E}} \cdot \frac{\delta_0}{W} \Rightarrow \left(\sigma = \frac{P}{A} + \frac{P}{1-P/P_E} \cdot \frac{\delta_0}{W}\right)\left(1-\frac{P}{P_E}\right)$$

$$\left(\sigma - \sigma\frac{P}{P_E} = \frac{P}{A} - \frac{P^2}{A \cdot P_E} + \frac{P\,\delta_0}{W}\right) \cdot A \cdot P_E$$

$$\sigma \cdot A \cdot P_E - \sigma \cdot A \cdot P = P \cdot P_E - P^2 + P \cdot \frac{A \cdot P_E \cdot \delta_0}{W}$$

$$\sigma_0 \cdot A \cdot P_E - \sigma_0 \cdot A \cdot P = P \cdot P_E - P^2 + P \cdot \frac{A \cdot P_E \cdot \delta_0}{W}$$

$$P^2 - \left[A \cdot \sigma_0 + P_E + \frac{A \cdot P_E \cdot \delta_0}{W} \right] \cdot P + \sigma_0 \cdot A \cdot P_E = 0$$

$A = 1635 \ mm^2$

$\sigma_0 = \sigma_{yield} = 275 \ MPa$

$P_E = 29 \ kN$

$\delta_0 = 5 \ mm$

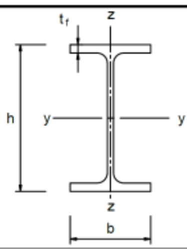
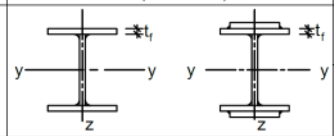
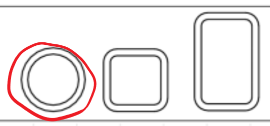
$W = 31.55 \times 10^3 \ mm^3$

Profile	Drawing	Ext. Diameter D [mm]	Wall thickness t [mm]	Weight m [kg/m]	External perimeter P [m]	Area A [mm²]	Shear area A_v [mm²]	Second moment of area I [×10⁶ mm⁴]	Radius of gyration i [mm]	Elastic section modulus W_{el} [×10³ mm³]	Plastic section modulus W_{pl} [×10³ mm³]
CHS 88.9 / 6.3	dxf	88.9	6.3	12.8	0.279	1635	1041	1.402	29.3	31.55	43.07

$$\rightarrow P^2 - \left[\underset{mm^2}{1635} \cdot \underset{MPa}{275} + \underset{(N)}{29000} + \frac{\overset{(mm^2)}{1635} \times \overset{(N)}{29000} \times \overset{(mm)}{5}}{31.55 \times 10^3 \ (mm^3)} \right] \cdot P + \underset{MPa}{275} \cdot \underset{(mm^2)}{1635} \cdot \underset{(N)}{29000} = 0$$

$$P = \begin{cases} 28491 \ N & = 28.5 \ kN \\ 457647 \ N & = 458 \ kN \end{cases} \longrightarrow \boxed{P = 28.5 \ kN}$$

Table 6.2: Selection of buckling curve for a cross-section

Cross section		Limits		Buckling about axis	Buckling curve S 235 S 275 S 355 S 420	Buckling curve S 460
Rolled sections		h/b > 1,2	$t_f \le 40$ mm	y – y z – z	a b	a_0 a_0
			40 mm < $t_f \le 100$	y – y z – z	b c	a a
		h/b ≤ 1,2	$t_f \le 100$ mm	y – y z – z	b c	a a
			t_f > 100 mm	y – y z – z	d d	c c
Welded I-sections			$t_f \le 40$ mm	y – y z – z	b c	b c
			t_f > 40 mm	y – y z – z	c d	c d
Hollow sections			hot finished	any	a	a_0
			cold formed	any	c	c

buckling curve a.

Table 5.1: Design values of initial local bow imperfection e_0 / L

Buckling curve acc. to Table 6.1	elastic analysis e_0 / L	plastic analysis e_0 / L
a_0	1 / 350	1 / 300
a	1 / 300	1 / 250
b	1 / 250	1 / 200
c	1 / 200	1 / 150
d	1 / 150	1 / 100

$$\frac{e_0}{L} = \frac{1}{300} \Rightarrow \boxed{e_0 = 16.67 \ mm} \rightarrow \delta_0$$

$$P^2 - \left[A \cdot \sigma + P_E + \frac{A \cdot P_E \cdot \delta_o}{W} \right] \cdot P + \sigma \cdot A \cdot P_E = 0$$

$\delta_o = 16.67\,mm$

$A = 1635\,mm^2$

$W = 31.55 \times 10^3\,mm^3$

$\sigma = 275\,Mpa$

$P_E = 29\,kN$

$$P = \begin{cases} 27.375\,kN \\ 476\,kN \quad X \end{cases} \rightarrow \boxed{27.375\,kN}$$

Profile	Drawing	Ext. Diameter D [mm]	Wall thickness t [mm]	Weight m [kg/m]	External perimeter P [m]	Area A [mm^2]	Shear area A_v [mm^2]	Second moment of area I [$\times 10^6\,mm^4$]	Radius of gyration i [mm]	Elastic section modulus W_{el} [$\times 10^3\,mm^3$]	Plastic section modulus W_{pl} [$\times 10^3\,mm^3$]
CHS 88.9 / 6.3	dxf	88.9	6.3	12.8	0.279	1635	1041	1.402	29.3	31.55	43.07

6.3.1.2 Buckling curves

(1) For axial compression in members the value of χ for the appropriate non-dimensional slenderness $\bar{\lambda}$ should be determined from the relevant buckling curve according to:

$$\frac{1}{8.65 + \sqrt{8.65^2 - 3.94^2}} = 0.061$$

$$\chi = \frac{1}{\Phi + \sqrt{\Phi^2 - \bar{\lambda}^2}} \quad \text{but } \chi \leq 1{,}0 \qquad (6.49)$$

where $\Phi = 0{,}5\left[1 + \alpha\left(\bar{\lambda} - 0{,}2\right) + \bar{\lambda}^2\right]$

$$0.5\left(1 + 0.21\left(3.94 - 0.2\right) + 3.94^2\right) = 8.65$$

$\bar{\lambda} = \sqrt{\dfrac{A f_y}{N_{cr}}}$ for Class 1, 2 and 3 cross-sections $= \sqrt{\dfrac{1635\,(mm^2) \cdot 275\,Mpa}{29000}} = 3.94$

$\bar{\lambda} = \sqrt{\dfrac{A_{eff} f_y}{N_{cr}}}$ for Class 4 cross-sections

α is an imperfection factor

N_{cr} is the elastic critical force for the relevant buckling mode based on the gross cross sectional properties.

(2) The imperfection factor α corresponding to the appropriate buckling curve should be obtained from Table 6.1 and Table 6.2.

Table 6.1: Imperfection factors for buckling curves

Buckling curve	a_0	a	b	c	d
Imperfection factor α	0,13	0,21	0,34	0,49	0,76

(3) The design buckling resistance of a compression member should be taken as:

$$N_{b,Rd} = \frac{\chi A f_y}{\gamma_{M1}} \quad \text{for Class 1, 2 and 3 cross-sections} \qquad (6.47)$$

$$N_{b,Rd} = \frac{\chi A_{eff} f_y}{\gamma_{M1}} \quad \text{for Class 4 cross-sections} \qquad (6.48)$$

where χ is the reduction factor for the relevant buckling mode.

$$N_{b,Rd} = \frac{\chi \cdot A \cdot f_y}{\gamma_{M1}} = \frac{0.061 \times 1635\,mm^2 \times 275\,Mpa}{1} = 27.43\,kN$$