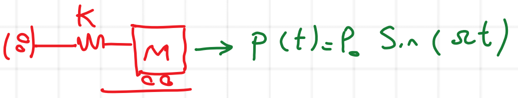
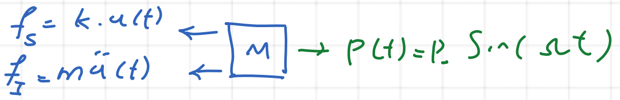
In our previous video, we explored the behavior of underdamped systems in free vibration, discussing their various characteristics and properties. In this video, we will be delving into a more detailed analysis of the response of a system under forced vibration with harmonic loading. We will be examining the effect of frequency of loading on the system's response. In the following videos, we will try to clarify the topic using examples.





$$\frac{m\ddot{u}(t) + k\,u(t) = P_0 \sin(\Omega t)}{m}$$

$$\ddot{u}(t) + \overbrace{\frac{k}{m}}^{\omega^2} u(t) = \frac{P_0}{m} \sin(\Omega t)$$

HE: $\ddot{u}(t) + \omega^2 u(t) = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm i\omega$

$$u_h(t) = A_1 \cos \omega t + A_2 \sin \omega t$$

if $(\omega \neq \Omega) \Rightarrow u_p(t) = B_1 \cos \Omega t + B_2 \sin \Omega t$

$$\dot{u}_p(t) = -B_1 \Omega \sin \Omega t + B_2 \Omega \cos \Omega t$$

$$\ddot{u}_p(t) = -B_1 \Omega^2 \cos \Omega t - B_2 \Omega^2 \sin \Omega t$$

$$\ddot{u}(t) + \omega^2 u(t) = \frac{P_0}{m} \sin \Omega t$$

$$-B_1 \Omega^2 \cos \Omega t + B_1 \omega^2 \cos \Omega t - B_2 \Omega^2 \sin \Omega t + B_2 \omega^2 \sin \Omega t = \frac{P_0}{m} \sin \Omega t$$

$$(-B_1 \Omega^2 + B_1 \omega^2) = 0 \qquad B_1 = 0 \qquad (\omega \neq \Omega)$$

$$-B_2 \Omega^2 + B_2 \omega^2 = \frac{P_0}{m} \Rightarrow B_2 = \frac{P_0}{m} \frac{1}{\omega^2 - \Omega^2}$$

$$\omega = \sqrt{\frac{k}{m}} \Rightarrow \omega^2 = \frac{k}{m} \Rightarrow m = \frac{k}{\omega^2} \Rightarrow B_2 = \frac{P_0 \omega^2}{k} \frac{1}{\omega^2 - \Omega^2} \Rightarrow B_2 = \frac{P_0}{k} \frac{1}{1 - \left(\frac{\Omega}{\omega}\right)^2}$$

$$\frac{\Omega}{\omega} = \beta$$

$$B_2 = \frac{P_0}{k} \frac{1}{1-\beta^2}$$

$$\boxed{\beta = \frac{\Omega}{\omega}}$$

$$u(t) = A_1 \cos\omega t + A_2 \sin\omega t + \frac{P_0}{k} \frac{1}{1-\beta^2} \sin\Omega t$$

$t=0 \rightarrow u(t=0) = u_0$

$t=0 \rightarrow \dot{u}(t=0) = \dot{u}_0$

$$\dot{u}(t) = -A_1 \omega \sin\omega t + A_2 \cdot \omega \cos\omega t + \frac{P_0}{k} \frac{\Omega}{1-\beta^2} \cos\Omega t$$

$u(t=0) = A_1 + 0 + 0 = u_0 \rightarrow \boxed{A_1 = u_0}$

$\dot{u}(t=0) = 0 + A_2 \omega + \frac{P_0}{k} \frac{\Omega}{1-\beta^2} = \dot{u}_0 \Rightarrow A_2 = \frac{\dot{u}_0}{\omega} - \frac{P_0}{k} \frac{\overset{\beta}{(\Omega/\omega)}}{1-\beta^2}$

$$u(t) = \underbrace{u_0 \cos\omega t + \left(\frac{\dot{u}_0}{\omega} - \frac{P_0}{k} \frac{\beta}{1-\beta^2}\right) \sin\omega t}_{\text{transient vibration}} + \underbrace{\frac{P_0}{k} \cdot \frac{1}{1-\beta^2} \sin\Omega t}_{\text{steady vibration}}$$